A two variable refined plate theory for orthotropic plate analysis

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Abstract

A new theory, which involves only two unknown functions and yet takes into account shear deformations, is presented for orthotropic plate analysis. Unlike any other theory, the theory presented gives rise to only two governing equations, which are completely uncoupled for static analysis, and are only inertially coupled (i.e., no elastic coupling at all) for dynamic analysis. Number of unknown functions involved is only two, as against three in case of simple shear deformation theories of Mindlin and Reissner. The theory presented is variationally consistent, has strong similarity with classical plate theory in many aspects, does not require shear correction factor, gives rise to transverse shear stress variation such that the transverse shear stresses vary parabolically across the thickness satisfying shear stress free surface conditions. Well studied examples, available in literature, are solved to validate the theory. The results obtained for plate with various thickness ratios using the theory are not only substantially more accurate than those obtained using the classical plate theory, but are almost comparable to those obtained using higher order theories having more number of unknown functions.

Keywords: Refined plate theory; Shear deformable plate theory; Orthotropic plate theory; Inertial coupling; Elastic coupling; Free vibrations

1. Introduction

The purpose of this paper is to present a new theory for orthotropic plate analysis, which involves only two unknown functions and yet takes into account shear deformations.

With the increasing use of composite materials, the need for advanced methods of analysis became obvious. In case of composite materials, transverse stresses and strains strongly influence the behavior. In particular, the transverse shear stress effects are more pronounced. The classical plate theory, which is not formulated to account for the effect of these stresses, is not satisfactorily applicable to orthotropic plate analysis. Therefore, over the years, researchers developed many theories, which took into account transverse shear effects.
It is worthwhile to note some developments in the plate theory.

Reissner (1944, 1945) was the first to develop a theory which incorporates the effect of shear. Reissner used stress-based approach. At the same level of approximation as the one utilized by Reissner, Mindlin (1951) employed displacement-based approach. As per Mindlin’s theory, transverse shear stress is assumed to be constant through the thickness of the plate, but this assumption violates the shear stress free surface conditions. Mindlin’s theory satisfies constitutive relations for transverse shear stresses and shear strains in an approximate manner by way of using shear correction factor. A good discussion about Reissner’s and Mindlin’s theories is available in the reference of Wang et al. (2001).

A refined plate theory for orthotropic plate, based on stress formulation, was proposed by Medwadowski (1958). In this theory, a nonlinear system of equations was derived from the corresponding equations of the three-dimensional theory of elasticity. The system of equations was linearized and use of stress function was made.

Approach of Librescu (1975) makes the use of weighted transverse displacement. Constitutive relations between shear stress and shear strain are satisfied. Reissner’s formulation comes out as a special case of Librescu’s approach.

Approach of Donnell (1976) is to make correction to the classical plate deflections. Donnell assumes uniform distribution of shear force across the thickness of the plate, and, to rectify the effects of the assumption, introduces a numerical factor, which needs to be adjusted.

Formulation by Levinson (1980) is based on displacement approach and his theory does not require shear correction factor. The governing equations for the motion of a plate obtained by Levinson’s approach are same as those obtained by Mindlin’s theory, provided that the shear correction factor associated with the Mindlin’s theory is taken as 5/6.


It is important to note that Srinivas et al. (1970) and Srinivas and Rao (1970) have presented a three-dimensional linear, small deformation theory of elasticity solution for static as well as dynamic analysis of isotropic, orthotropic and laminated simply-supported rectangular plates.

It may be worthwhile to note that a critical review of plate theories was carried out by Vasil’ev (1992). Whereas, Liew et al. (1995) surveyed plate theories particularly applied to thick plate vibration problems. A recent review paper is by Ghugal and Shimpi (2002).

Plate theories can be developed by expanding the displacements in power series of the coordinate normal to the middle plane. In principle, theories developed by this means can be made as accurate as desired simply by including a sufficient number of terms in the series. These higher-order theories are cumbersome and computationally more demanding, because each additional power of the thickness coordinate, an additional dependent is introduced into the theory.

It has been noted by Lo et al. (1977a) that due to the higher order of terms included in their theory, the theory is not convenient to use. This observation is more or less true for many other higher order theories as well. And, thus there is a scope to develop simple to use higher order plate theory.

It is to be noted that Shimpi (2002) presented a theory for isotropic plates. He applied the theory to flexure of shear-deformable isotropic plates. The theory, using only two unknown functions, gave rise to two governing equations, which are uncoupled in case of static analysis. Also, unlike any other theory, the theory has strong similarities with the classical plate theory in some aspects. In this paper, using similar approach, a new orthotropic plate theory is presented.

2. Orthotropic plate under consideration

Consider a plate (of length $a$, width $b$, and thickness $h$). The plate occupies (in $0-x-y-z$ right-handed Cartesian coordinate system) a region

$$0 \leq x \leq a; \quad 0 \leq y \leq b; \quad -h/2 \leq z \leq h/2$$

(1)
The plate can have any meaningful boundary conditions at edges $x = 0$, $x = a$ and $y = 0$, $y = b$. The orthotropic plate has following material properties: $E_1$, $E_2$ are elastic moduli, $G_{12}$, $G_{23}$, $G_{31}$ are shear moduli and $\mu_{12}$, $\mu_{21}$ are Poisson’s ratios.

Here subscripts 1, 2, 3 correspond to $x$, $y$, $z$ directions of Cartesian coordinate system, respectively (this notation is the same as that used by Jones (1999, Chapter 2) and Reddy (1984)).

For isotropic plate these above material properties reduce to $E_1 = E_2 = E$, $G_{12} = G_{23} = G_{31} = G$, $\mu_{12} = \mu_{21} = \mu$.

3. Refined plate theory (RPT) for orthotropic plate

3.1. Assumptions of RPT

Assumptions of RPT would be as follows:

1. The displacements ($u$ in $x$-direction, $v$ in $y$-direction, $w$ in $z$-direction) are small in comparison with the plate thickness and, therefore, strains involved are infinitesimal. As a result, normal strains $\epsilon_x$, $\epsilon_y$, $\epsilon_z$ and shear strains $\gamma_{xy}$, $\gamma_{yz}$, $\gamma_{zx}$ can be expressed in terms of displacements $u$, $v$, $w$ by using strain-displacement relations:

$$
\begin{align*}
\epsilon_x &= \frac{\partial u}{\partial x}; \quad \epsilon_y = \frac{\partial v}{\partial y}; \quad \epsilon_z = \frac{\partial w}{\partial z} \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}; \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}; \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}
\end{align*}
$$

2. The transverse displacement $w$ has two components: bending component $w_b$ and shear component $w_s$. Both the components are functions of coordinates $x$, $y$ and time $t$ only.

$$w(x, y, t) = w_b(x, y, t) + w_s(x, y, t)$$

3. (a) In general, transverse normal stress $\sigma_z$ is negligible in comparison with inplane stresses $\sigma_x$ and $\sigma_y$. Therefore, stresses $\sigma_x$ and $\sigma_y$ are related to strains $\epsilon_x$ and $\epsilon_y$ by the following constitutive relations:

$$
\begin{align*}
\sigma_x &= \frac{E_1}{1 - \mu_{12}\mu_{21}} \epsilon_x + \frac{\mu_{12}E_2}{1 - \mu_{12}\mu_{21}} \epsilon_y \\
\sigma_y &= \frac{E_2}{1 - \mu_{12}\mu_{21}} \epsilon_y + \frac{\mu_{21}E_1}{1 - \mu_{12}\mu_{21}} \epsilon_x
\end{align*}
$$

(b) The shear stresses $\tau_{xy}$, $\tau_{yz}$, $\tau_{zx}$ are related to shear strains $\gamma_{xy}$, $\gamma_{yz}$, $\gamma_{zx}$ by the following constitutive relations:

$$
\begin{align*}
\tau_{xy} &= G_{12}\gamma_{xy}; \quad \tau_{yz} = G_{23}\gamma_{yz}; \quad \tau_{zx} = G_{31}\gamma_{zx}
\end{align*}
$$
4. The displacement \( u \) in \( x \)-direction consists of bending component \( u_b \) and shear component \( u_s \). Similarly, the displacement \( v \) in \( y \)-direction consists of bending component \( v_b \) and shear component \( v_s \).

\[
\begin{align*}
\text{\( u \)} &= \text{\( u_b \)} + \text{\( u_s \)}; \\
\text{\( v \)} &= \text{\( v_b \)} + \text{\( v_s \)}
\end{align*}
\]  \hspace{1cm} (6)

(a) The bending component \( u_b \) of displacement \( u \) and \( v_b \) of displacement \( v \) are assumed to be analogous, respectively, to the displacements \( u \) and \( v \) given by classical plate theory (CPT). Therefore, the expressions for \( u_b \) and \( v_b \) can be given as

\[
\begin{align*}
\text{\( u_b \)} &= -z \frac{\partial w_b}{\partial x} \\
\text{\( v_b \)} &= -z \frac{\partial w_b}{\partial y}
\end{align*}
\]  \hspace{1cm} (7)

(b) The shear component \( u_s \) of displacement \( u \) and the shear component \( v_s \) of displacement \( v \) are such that

(i) they give rise, in conjunction with \( w_s \), to the parabolic variations of shear strains \( \gamma_{yz}, \gamma_{zx} \) and therefore to shear stresses \( \tau_{zx} \) and \( \tau_{yz} \),

(ii) their contribution toward strains \( \varepsilon_x, \varepsilon_y, \) and \( \gamma_{xy} \) is such that in the moments \( M_x, M_y, \) and \( M_{xy} \) there is no contribution from the components \( u_s \) and \( v_s \).

3.2. Displacements, moments, shear forces in RPT

Based on the assumptions made in the preceding section, it is possible, with some efforts, to get the expressions for the shear component \( u_s \) of displacement \( u \), and shear component \( v_s \) of displacement \( v \); and these can be written as

\[
\begin{align*}
\text{\( u_s \)} &= h \left[ \frac{1}{4} \left( \frac{z}{h} \right) - \frac{5}{3} \left( \frac{z}{h} \right)^3 \right] \frac{\partial^2 w_s}{\partial x^2} \\
\text{\( v_s \)} &= h \left[ \frac{1}{4} \left( \frac{z}{h} \right) - \frac{5}{3} \left( \frac{z}{h} \right)^3 \right] \frac{\partial^2 w_s}{\partial y^2}
\end{align*}
\]  \hspace{1cm} (9)

Using expressions (3), and (6)–(10), one can write expressions for displacements \( u, v, w \) as

\[
\begin{align*}
\text{\( u(x, y, z, t) \)} &= -z \frac{\partial w_b}{\partial x} + h \left[ \frac{1}{4} \left( \frac{z}{h} \right) - \frac{5}{3} \left( \frac{z}{h} \right)^3 \right] \frac{\partial^2 w_b}{\partial x^2} \\
\text{\( v(x, y, z, t) \)} &= -z \frac{\partial w_b}{\partial y} + h \left[ \frac{1}{4} \left( \frac{z}{h} \right) - \frac{5}{3} \left( \frac{z}{h} \right)^3 \right] \frac{\partial^2 w_b}{\partial y^2} \\
\text{\( w(x, y, t) \)} &= w_b(x, y, t) + w_s(x, y, t)
\end{align*}
\]  \hspace{1cm} (11–13)

Using expressions for displacements (11)–(13) in strain-displacement relations (2), the expressions for strains can be obtained.

\[
\begin{align*}
\varepsilon_x &= -z \frac{\partial^2 w_b}{\partial x^2} + h \left[ \frac{1}{4} \left( \frac{z}{h} \right) - \frac{5}{3} \left( \frac{z}{h} \right)^3 \right] \frac{\partial^2 w_b}{\partial x^2} \\
\varepsilon_y &= -z \frac{\partial^2 w_b}{\partial y^2} + h \left[ \frac{1}{4} \left( \frac{z}{h} \right) - \frac{5}{3} \left( \frac{z}{h} \right)^3 \right] \frac{\partial^2 w_b}{\partial y^2} \\
\varepsilon_z &= 0 \\
\gamma_{xy} &= -2z \frac{\partial^2 w_b}{\partial x \partial y} + 2h \left[ \frac{1}{4} \left( \frac{z}{h} \right) - \frac{5}{3} \left( \frac{z}{h} \right)^3 \right] \frac{\partial^2 w_b}{\partial x \partial y}
\end{align*}
\]  \hspace{1cm} (14–17)
\[ \gamma_{xz} = \frac{5}{4} - \frac{5}{h^2} \left( \frac{w'}{y} \right)^2 \]  
\[ \gamma_{zx} = \frac{5}{4} - \frac{5}{h^2} \left( \frac{w}{x} \right)^2 \]  

Expressions for stresses can be obtained using strain expressions from (14)–(19) in constitutive relations (4) and (5). Constitutive relation can be written in matrix form as follows:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 & 0 & 0 \\
Q_{12} & Q_{22} & 0 & 0 & 0 \\
0 & 0 & Q_{66} & 0 & 0 \\
0 & 0 & 0 & Q_{44} & 0 \\
0 & 0 & 0 & 0 & Q_{55}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix}
\]

where,

\[
Q_{11} = \frac{E_1}{1 - \mu_{12}\mu_{21}}; \quad Q_{12} = \frac{\mu_{12}E_2}{1 - \mu_{12}\mu_{21}} = \frac{\mu_{21}E_1}{1 - \mu_{12}\mu_{21}} \\
Q_{22} = \frac{E_2}{1 - \mu_{12}\mu_{21}}; \quad Q_{44} = G_{23}; \quad Q_{55} = G_{31}; \quad Q_{66} = G_{12}
\]

It is well known that the material property matrix in expression (20) is symmetric (e.g. reference of Jones, 1999).

The moments and shear forces are defined as

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy} \\
Q_x \\
Q_y
\end{bmatrix} =
\int_{z=-h/2}^{z=+h/2}
\begin{bmatrix}
\sigma_x z \\
\sigma_y z \\
\tau_{xy} z \\
\tau_{xz} \\
\tau_{yz}
\end{bmatrix}
\, dz
\]

Using expressions (14)–(20) in (22), expressions for moments \(M_x, M_y, M_{xy}\) and shear forces \(Q_x\) and \(Q_y\) can be obtained. These expressions are:

\[
M_x = -D_{11} \frac{\partial^2 w_b}{\partial x^2} + D_{12} \frac{\partial^2 w_b}{\partial x \partial y} \]
\[
M_y = -D_{22} \frac{\partial^2 w_b}{\partial y^2} + D_{12} \frac{\partial^2 w_b}{\partial x \partial y} \]
\[
M_{xy} = -2D_{66} \frac{\partial^2 w_b}{\partial x \partial y} \]
\[
Q_x = A_{55} \frac{\partial w_z}{\partial x} \]
\[
Q_y = A_{44} \frac{\partial w_z}{\partial y} \]

where,

\[
D_{11} = \frac{Q_{11} h^3}{12}; \quad D_{22} = \frac{Q_{22} h^3}{12}; \quad D_{12} = \frac{Q_{12} h^3}{12}; \quad D_{66} = \frac{Q_{66} h^3}{12}; \\
A_{44} = \frac{5Q_{44} h^3}{6}; \quad A_{55} = \frac{5Q_{55} h^3}{6}
\]
It may be noted that expressions for moments \( M_x, M_y \) and \( M_{xy} \) contain only \( w_b \) as an unknown function. Also, the expressions for shear forces \( Q_x \) and \( Q_y \) contain only \( w_s \) as an unknown function.

3.3. Expressions for kinetic and total potential energies

It should be noted that displacement \( w \), given by Eq. (13), is not a function of \( z \). As a result of this, normal strain \( \varepsilon_z \) comes out to be zero. Therefore, the expressions for kinetic energy \( T \) and total potential energy \( II \) of the plate can be written as

\[
T = \int_{x=-\frac{a}{2}}^{x=a} \int_{y=0}^{y=b} \int_{z=-\frac{1}{2}}^{z=1} \frac{1}{2} \rho \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \, dx \, dy \, dz
\]

(29)

\[
II = \int_{x=-\frac{a}{2}}^{x=a} \int_{y=0}^{y=b} \int_{z=-\frac{1}{2}}^{z=1} \frac{1}{2} \left[ \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy} + \tau_{z} \gamma_{xz} \right] \, dx \, dy \, dz
- \int_{y=0}^{y=b} \int_{x=0}^{x=a} q[w_b + w_s] \, dx \, dy
\]

(30)

in which \( q \) is the intensity of externally applied transverse load (acting along the \( z \)-direction). Using expressions (11)–(20) in Eqs. (29) and (30), expressions for kinetic energy and total potential energy can be written as

\[
T = \frac{\rho h^3}{24} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ \left[ \frac{\partial}{\partial t} \left( \frac{\partial w_b}{\partial x} \right) \right]^2 + \left[ \frac{\partial}{\partial t} \left( \frac{\partial w_b}{\partial y} \right) \right]^2 \right\} \, dx \, dy + \frac{\rho h^3}{2016} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ \left[ \frac{\partial}{\partial t} \frac{\partial w_b}{\partial x} \right]^2 \right. \\
+ \left[ \frac{\partial}{\partial t} \frac{\partial w_s}{\partial y} \right]^2 \right\} \, dx \, dy
\]

(31)

\[
II = \frac{1}{2} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ D_{11} \frac{\partial^2 w_b}{\partial x^2} + D_{22} \frac{\partial^2 w_b}{\partial y^2} \right\} + 2D_{12} \frac{\partial^2 w_b}{\partial x \partial y} + 4D_{66} \frac{\partial^2 w_b}{\partial x \partial y} \right\} \, dx \, dy + \frac{1}{2} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ \frac{1}{84} D_{11} \frac{\partial^2 w_s}{\partial x^2} + D_{22} \frac{\partial^2 w_s}{\partial y^2} \right\} + 2D_{12} \frac{\partial^2 w_s}{\partial x \partial y} + 4D_{66} \frac{\partial^2 w_s}{\partial x \partial y} \right\} \, dx \, dy \\
+ \frac{1}{2} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ A_{44} \frac{\partial^2 w_b}{\partial x^2} + A_{55} \frac{\partial^2 w_s}{\partial x^2} \right\} \, dx \, dy - \int_{y=0}^{y=b} \int_{x=0}^{x=a} q[w_b + w_s] \, dx \, dy
\]

(32)

3.4. Obtaining governing equations and boundary conditions in RPT by using Hamilton’s principle

Governing differential equations and boundary conditions can be obtained using well known Hamilton’s principle

\[
\int_{t_1}^{t_2} \delta(T - II) \, dt = 0
\]

(33)

where \( \delta \) indicates a variation w.r.t. \( x \) and \( y \) only; \( t_1, t_2 \) are values of time variable \( t \) at the start and at the end of time interval (in the context of Hamilton’s Principle), respectively.

Using expressions (31) and (32) in the preceding equation and integrating the equation by parts, taking into account the independent variations of \( w_b \) and \( w_s \), one obtains the governing differential equations and boundary conditions, and these are as follows:
3.4.1. Governing equations in RPT

The governing differential equations for the plate are

\[
D_{11} \frac{\partial^4 w_b}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_b}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_b}{\partial y^4} - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \left( \nabla^2 w_b \right) + \rho h \left( \frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_b}{\partial t^2} \right) = q
\]  
(34)

\[
- \left[ A_{55} \frac{\partial^2 w_s}{\partial x^2} + A_{44} \frac{\partial^2 w_s}{\partial y^2} \right] + \frac{1}{84} \left[ D_{11} \frac{\partial^4 w_s}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_s}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_s}{\partial y^4} - \frac{\rho h^3}{84 \times 12} \frac{\partial^2}{\partial t^2} \left( \nabla^2 w_s \right) \right] + \rho h \left( \frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_b}{\partial t^2} \right) = q
\]  
(35)

where,

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]  
(36)

3.4.2. Boundary conditions in RPT

The boundary conditions of the plate are given as follows:

1. At corners \((x = 0, y = 0), (x = 0, y = b), (x = a, y = 0),\) and \((x = a, y = b),\) the following conditions hold:
   (a) The condition involving \(w_b\) (i.e., bending component of transverse displacement)

\[
(2D_{66}) \left[ \frac{\partial^2 w_b}{\partial x \partial y} \right] = 0 \quad \text{or} \quad w_b \text{ is specified}
\]  
(37)

(b) The condition involving \(w_s\) (i.e., shear component of transverse displacement)

\[
\frac{1}{84} (2D_{66}) \left[ \frac{\partial^2 w_s}{\partial x \partial y} \right] = 0 \quad \text{or} \quad w_s \text{ is specified}
\]  
(38)

2. On edges \(x = 0\) and \(x = a,\) the following conditions hold:
   (a) The conditions involving \(w_b\) (i.e., bending component of transverse displacement)

\[
- \left[ D_{11} \frac{\partial^3 w_b}{\partial x^3} + (D_{12} + 4D_{66}) \frac{\partial^3 w_b}{\partial x^2 \partial y} \right] + \frac{\rho h^3}{12} \left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_b}{\partial x} \right) \right] = 0 \quad \text{or} \quad w_b \text{ is specified}
\]  
(39)

\[
- \left[ D_{11} \frac{\partial^3 w_b}{\partial x^3} + D_{22} \frac{\partial^3 w_b}{\partial y^3} \right] = 0 \quad \text{or} \quad \frac{\partial w_b}{\partial x} \text{ is specified}
\]  
(40)

(b) The conditions involving \(w_s\) (i.e., shear component of transverse displacement)

\[
A_{55} \frac{\partial^2 w_s}{\partial x} - \frac{1}{84} \left[ D_{11} \frac{\partial^3 w_s}{\partial x^3} + (D_{12} + 4D_{66}) \frac{\partial^3 w_s}{\partial x^2 \partial y} + \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_s}{\partial x} \right) \right] = 0 \quad \text{or} \quad w_s \text{ is specified}
\]  
(41)

\[
- \frac{1}{84} \left[ D_{11} \frac{\partial^2 w_s}{\partial x^2} + D_{22} \frac{\partial^2 w_s}{\partial y^2} \right] = 0 \quad \text{or} \quad \frac{\partial w_s}{\partial x} \text{ is specified}
\]  
(42)

3. On edges \(y = 0\) and \(y = b,\) the following conditions hold:
   (a) The conditions involving \(w_b\) (i.e., bending component of transverse displacement)

\[
- \left[ D_{22} \frac{\partial^3 w_b}{\partial y^3} + (D_{12} + 4D_{66}) \frac{\partial^3 w_b}{\partial x \partial y^2} + \frac{\rho h^3}{12} \left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial w_b}{\partial y} \right) \right] \right] = 0 \quad \text{or} \quad w_b \text{ is specified}
\]  
(43)

\[
- \left[ D_{22} \frac{\partial^2 w_b}{\partial y^2} + D_{22} \frac{\partial^2 w_b}{\partial x^2} \right] = 0 \quad \text{or} \quad \frac{\partial w_b}{\partial y} \text{ is specified}
\]  
(44)
The conditions involving $w_s$ (i.e., shear component of transverse displacement)

$$A_{44} \frac{\partial w_s}{\partial y} - \frac{1}{84} \left[ D_{22} \frac{\partial^3 w_s}{\partial y^3} + (D_{12} + 4D_{66}) \frac{\partial^3 w_s}{\partial x^2 \partial y} + \frac{\rho h^2}{12} \frac{\partial^2}{\partial y^2} \left( \frac{\partial w_s}{\partial y} \right) \right] = 0 \quad \text{or} \quad w_s \text{ is specified} \quad (45)$$

$$- \frac{1}{84} \left[ D_{22} \frac{\partial^2 w_s}{\partial y^2} + D_{12} \frac{\partial^2 w_s}{\partial x^2} \right] = 0 \quad \text{or} \quad \frac{\partial w_s}{\partial y} \text{ is specified} \quad (46)$$

4. Comments on RPT

1. With respect to governing equations, following can be noted:
   (a) In RPT, there are two governing equations (Eqs. (34) and (35)). Both the governing equations are fourth-order partial differential equations.
   (b) The governing equations involve only two unknown functions (i.e., bending component $w_b$ and shear component $w_s$ of transverse deflection). Even theories of Reissner (1944), Mindlin (1951), which are first order shear deformation theories and are considered to be simple ones, involve three unknown functions. Mindlin’s theory does not exactly satisfy transverse shear stresses and shear strains constitutive relations. Whereas, in contrast, in RPT, these constitutive relations are exactly satisfied.
   (c) It is emphasized here that RPT is the only theory, to the best knowledge of the authors, wherein in the governing equations, there is only inertial coupling, and there is no elastic coupling at all.

2. With respect to boundary conditions, following can be noted:
   (a) There are two conditions per corner.
      (i) One condition is stated in terms of $w_b$ and its derivatives only (i.e., condition (37)).
      (ii) The second condition is stated in terms of $w_s$ and its derivatives only (i.e., condition (38)).
   (b) There are four boundary conditions per edge.
      (i) Two conditions are stated in terms of $w_b$ and its derivatives only (e.g., in the case of edge $x = 0$, conditions (39) and (40)).
      (ii) The remaining two conditions are stated in terms of $w_s$ and its derivatives only (e.g., in the case of edge $x = 0$, conditions (41) and (42)).

3. Some entities of RPT (e.g., a governing equation, moment expressions, boundary conditions) have strong similarity with those of CPT. The relevant equations, expressions in respect of CPT are given in Jones (1999, Chapter 5).
   (a) The following entities of RPT are identical, save for the appearance of the subscript, to the corresponding entities of the CPT:
      (i) Moment expressions for $M_x$, $M_y$, $M_{xy}$ (i.e., expressions (23)–(25)).
      (ii) Corner boundary condition (i.e., condition (37)).
      (iii) Edge boundary conditions (i.e., conditions (39), (40), (43) and (44)).
   The bending component $w_b$ of transverse displacement figures in the just mentioned entities of RPT, whereas transverse displacement $w$ figures in the corresponding equations of the CPT.
   (b) One of the two governing equations (i.e., Eq. (34)) has strong similarity with the governing equation of CPT. (If in Eq. (34) the entity $\frac{\partial w_s}{\partial y}$ is ignored, and if $w_b$ is replaced by $w$, then the resulting equation is identical to the governing equation of CPT.)

4. It is seen from the displacements $u$, $v$, $w$ expressions (11)–(13) that, as the thickness becomes smaller and smaller, the theory converges in the limit to the classical plate theory and, therefore, it is equally applicable to thick as well as thin plate analysis.

5. Illustrative examples for static analysis of simply supported rectangular plates

Well studied examples, available in literature, would be taken up to demonstrate the effectiveness of RPT. The examples have also been studied by Reddy (1984), Srinivas and Rao (1970).

Consider a plate (of length $a$, width $b$, and thickness $h$) of orthotropic material. The plate occupies (in $0–x–y–z$ right-handed Cartesian coordinate system) a region defined by Eq. (1). The plate has simply supported
boundary conditions at all four edges \( x = 0, \ x = a, \ y = 0 \) and \( y = b \). The plate carries on surface \( z = -h/2 \), a uniformly distributed load of intensity \( q_0 \) acting in the \( z \)-direction. The properties of orthotropic material used for analysis are indicated in the concerned result Tables 1–7.

5.1. Governing equations for illustrative examples (static analysis)

The governing equations for static analysis of plate can be obtained from Eqs. (34) and (35) by setting the kinetic energy terms to zero, as follows:

\[
D_{11} \frac{\partial^4 w_b}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_b}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_b}{\partial y^4} = q_0 \\
- \left[ A_{55} \frac{\partial^2 w_s}{\partial x^2} + A_{44} \frac{\partial^2 w_s}{\partial y^2} \right] + \frac{1}{84} \left[ D_{11} \frac{\partial^4 w_s}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_s}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_s}{\partial y^4} \right] = q_0
\]

5.2. Boundary conditions for illustrative examples (static analysis)

The boundary conditions of the plate are given as follows:

1. At corners \( (x = 0, \ y = 0), (x = 0, \ y = b), (x = a, \ y = 0), \) and \( (x = a, \ y = b) \) the following conditions hold:

\[
w_b = 0 \\
w_s = 0
\]

2. On edges \( x = 0 \) and \( x = a \), the following conditions hold:

\[
w_b = 0 \\
- \left[ D_{11} \frac{\partial^2 w_b}{\partial x^2} + D_{12} \frac{\partial^2 w_b}{\partial y^2} \right] = 0 \\
w_s = 0 \\
- \frac{1}{84} \left[ D_{11} \frac{\partial^2 w_s}{\partial x^2} + D_{12} \frac{\partial^2 w_s}{\partial y^2} \right] = 0
\]

3. On edges \( y = 0 \) and \( y = b \), the following conditions hold:

\[
w_b = 0 \\
- \left[ D_{22} \frac{\partial^2 w_b}{\partial y^2} + D_{12} \frac{\partial^2 w_b}{\partial x^2} \right] = 0 \\
w_s = 0 \\
- \frac{1}{84} \left[ D_{22} \frac{\partial^2 w_s}{\partial y^2} + D_{12} \frac{\partial^2 w_s}{\partial x^2} \right] = 0
\]

5.3. Solution of illustrative examples (static analysis)

The following displacement functions \( w_b \) and \( w_s \) satisfy the boundary conditions (49)–(58).

\[
w_b = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} W_{bm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \\
w_s = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} W_{sm} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right)
\]

where, \( W_{bm}, W_{sm} \) are constant coefficients associated with \( w_b \) and \( w_s \), respectively.
Also externally applied uniformly distributed load of intensity \( q_0 \) can be represented in a double Fourier series as follows:

\[
q_0 = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} q_{mn} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right)
\]

where

\[
q_{mn} = \frac{16q_0}{\pi^2 mn} \quad \text{for uniformly distributed load of intensity } q_0
\]

Using expressions (59)–(61) in the governing Eqs. (47) and (48), one obtains two completely uncoupled equations which can be written in matrix form, as follows:

\[
\begin{bmatrix}
K_{11mn} & 0 \\
0 & K_{22mn}
\end{bmatrix}
\begin{bmatrix}
W_{bmn} \\
W_{swn}
\end{bmatrix}
= \begin{bmatrix}
q_{mn} \\
q_{mn}
\end{bmatrix}
\]

where,

\[
K_{11mn} = D_{11} \left( \frac{m \pi}{a} \right)^4 + 2(D_{12} + 2D_{66}) \left( \frac{m \pi}{a} \right)^2 \left( \frac{n \pi}{b} \right)^2 + D_{22} \left( \frac{n \pi}{b} \right)^4
\]

\[
K_{22mn} = A_{55} \left( \frac{m \pi}{a} \right)^2 + A_{44} \left( \frac{n \pi}{b} \right)^2 + \frac{1}{84} \left[ D_{11} \left( \frac{m \pi}{a} \right)^4 + 2(D_{12} + 2D_{66}) \left( \frac{m \pi}{a} \right)^2 \left( \frac{n \pi}{b} \right)^2 + D_{22} \left( \frac{n \pi}{b} \right)^4 \right]
\]

Coefficients \( W_{bmn}, W_{swn} \) can be determined from Eq. (62), and then can be substituted in Eqs. (59) and (60) to get displacements.

Here it is observed from the Eq. (62) that bending and shearing components are uncoupled.

The following nondimensionalized deflections \( \bar{w}, \bar{w} \) and stresses \( \sigma_x, \sigma_y, \tau_{zx} \) are tabulated in Tables 1–5.

\[
\bar{w} = wQ_{11}/hq_0; \quad \text{for orthotropic plate}
\]

\[
\bar{w} = wG/hq_0; \quad \text{for isotropic plate and}
\]

\[
\bar{\sigma}_x = \sigma_x/q_0
\]

\[
\bar{\sigma}_y = \sigma_y/q_0
\]

\[
\bar{\tau}_{zx} = \tau_{zx}/q_0; \quad \text{for orthotropic plate.}
\]

Results are discussed separately after Table 7.

### Table 1
Comparison of nondimensional central displacement \( \bar{w} \) of simply-supported orthotropic rectangular plate under uniformly distributed transverse load

<table>
<thead>
<tr>
<th>Plate dimensional parameters</th>
<th>Nondimensional displacement ( \bar{w} ) at ( x = a/2, y = b/2 ) by various theories</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a/b )</td>
<td>( h/a )</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>0.5</td>
<td>0.05</td>
</tr>
<tr>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>0.14</td>
<td>0.07</td>
</tr>
<tr>
<td>1.0</td>
<td>0.05</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>0.14</td>
<td>0.14</td>
</tr>
<tr>
<td>2.0</td>
<td>0.05</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>0.14</td>
<td>0.28</td>
</tr>
</tbody>
</table>

\( \bar{w} = wQ_{11}/hq_0; \) (\( E_2/E_1 = 0.52500, G_{12}/E_1 = 0.26293, G_{13}/E_1 = 0.15991, G_{23}/E_1 = 0.26681, \mu_{12} = 0.44046, \mu_{21} = 0.23124 \)).

\( \text{b} \) Taken from reference of Srinivas and Rao (1970).

\( \text{c} \) Taken from reference of Reddy (1984).
6. Illustrative examples for free vibrations of simply supported rectangular plates

Well studied examples, available in literature, would be taken up to demonstrate the effectiveness of RPT. The examples have also been studied by Reddy (1984, 1985), Srinivas et al. (1970), Srinivas and Rao (1970).

Consider a plate (of length \(a\), width \(b\), and thickness \(h\)) of orthotropic material. The plate occupies (in \(0 - x - y - z\) right-handed Cartesian coordinate system) a region defined by Eq. (1). The plate has simply supported boundary conditions at all four edges \(x = 0, x = a, y = 0\) and \(y = b\).

The properties of orthotropic material used for analysis are indicated in the concerned result Tables 1–7.

6.1. Governing equations for illustrative examples (free vibration analysis)

The governing equations for free vibration of plate can be obtained from Eqs. (34) and (35) by setting the external load (i.e., transverse load \(q\)) to zero, as follows:

<table>
<thead>
<tr>
<th>Plate dimensional parameters</th>
<th>Non-dimensional displacement (\bar{\omega}) at (x = a/2, y = b/2) by various theories</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a/b) (h/a) (h/b)</td>
<td>EXACT (^{b})</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>0.5</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
</tr>
<tr>
<td>1.0</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
</tr>
<tr>
<td>2.0</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
</tr>
</tbody>
</table>

Note: Results are not available for Reddy’s theory.

\(^{a}\) \(\bar{\omega} = wG/hq_0\) (\(E_2 = E_1\), \(G_{12} = G_{13} = G_{23} = G = [E/2(1 + \mu)]\), \(\mu_{12} = \mu_{21} = \mu = 0.3\).

\(^{b}\) Taken from reference of Srinivas (1970).

Table 3

Comparison of nondimensional stress \(\bar{\sigma}\) of simply-supported orthotropic rectangular plate (under uniformly distributed transverse load)\(^{b}\)

<table>
<thead>
<tr>
<th>Plate dimensional parameters</th>
<th>Nondimensional stress (\bar{\sigma}) at (x = a/2, y = b/2, z = h/2) by various theories</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a/b) (h/a) (h/b)</td>
<td>EXACT (^{b})</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>0.5</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
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<tr>
<td>1.0</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
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<tr>
<td></td>
<td>0.14</td>
</tr>
<tr>
<td>2.0</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
</tr>
</tbody>
</table>

\(^{a}\) At \(x = a/2, y = b/2, z = h/2\) \(\bar{\sigma} = \sigma_3/q_0\) (\(E_2/E_1 = 0.52500\), \(G_{12}/E_1 = 0.26293\), \(G_{13}/E_1 = 0.15991\), \(G_{23}/E_1 = 0.26681\), \(\mu_{12} = 0.44046\), \(\mu_{23} = 0.23124\)).

\(^{b}\) Taken from reference of Srinivas and Rao (1970).

\(^{c}\) Taken from reference of Reddy (1984).
\[ D_{11} \frac{\partial^4 w_b}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_b}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_b}{\partial y^4} + \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} (\nabla^2 w_b) + \rho h^2 (w_b + w_s) = 0 \] (63)

\[ - \left[ A_{55} \frac{\partial^2 w_s}{\partial x^2} + A_{44} \frac{\partial^2 w_s}{\partial y^2} \right] + \frac{1}{84} \left[ D_{11} \frac{\partial^4 w_s}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w_s}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_s}{\partial y^4} \right] - \frac{\rho h^3}{84 \times 12} \frac{\partial^2}{\partial t^2} (\nabla^2 w_s) + \rho h^2 (w_b + w_s) = 0 \] (64)

6.2. Boundary conditions for illustrative examples (free vibration analysis)

Boundary conditions are as those of illustrative example I (i.e., expressions (49)–(58)).

Table 4

<table>
<thead>
<tr>
<th>Plate dimensional parameters</th>
<th>Nondimensional stress (\sigma_{x} ) at (x = a/2, y = b/2, z = h/2) by various theories</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a/b)</td>
<td>(h/a)</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>0.5</td>
<td>0.05</td>
</tr>
<tr>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>0.14</td>
<td>0.07</td>
</tr>
<tr>
<td>1.0</td>
<td>0.05</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>0.14</td>
<td>0.14</td>
</tr>
</tbody>
</table>

Note: Results are not available for Reddy's theory.

\(^a\) At \(x = a/2, y = b/2, z = h/2\) \(\sigma_{x} = \sigma_{y}/q_{0}\) \((E_2/E_1 = 0.52500, G_{12}/E_1 = 0.26293, G_{13}/E_1 = 0.15991, G_{23}/E_1 = 0.26681, \mu_{12} = 0.44046, \mu_{13} = 0.23124)\).

\(^b\) Taken from reference of Srinivas and Rao (1970).

Table 5

<table>
<thead>
<tr>
<th>Plate dimensional parameters</th>
<th>Nondimensional stress (\tau_{zx}) at (x = 0, y = b/2, z = h/2) by various theories</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a/b)</td>
<td>(h/a)</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>0.5</td>
<td>0.05</td>
</tr>
<tr>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>0.14</td>
<td>0.07</td>
</tr>
<tr>
<td>1.0</td>
<td>0.05</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>0.14</td>
<td>0.14</td>
</tr>
</tbody>
</table>

Note: Results are not available for Reddy's theory.

\(^a\) At \(x = 0, y = b/2, z = h/2\) \(\tau_{zx} = \tau_{sy}/q_{0}\) \((E_2/E_1 = 0.52500, G_{12}/E_1 = 0.26293, G_{13}/E_1 = 0.15991, G_{23}/E_1 = 0.26681, \mu_{12} = 0.44046, \mu_{13} = 0.23124)\).

\(^b\) Taken from reference of Srinivas and Rao (1970).

\(^c\) Taken from reference of Reddy (1984).
6.3. Solution of illustrative examples (free vibration analysis)

The following displacement functions \( w_b \) and \( w_s \) satisfy the boundary conditions (49)–(58):

\[
w_b = \sum_{m=1,2,...}^\infty \sum_{n=1,2,...}^\infty W_{bmn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin(\omega_{mnt})
\]

\[
w_s = \sum_{m=1,2,...}^\infty \sum_{n=1,2,...}^\infty W_{smn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin(\omega_{mnt})
\]

where, \( W_{bmn} \), \( W_{smn} \) are constant coefficients and \( \omega_{mnt} \) is the circular frequency of vibration associated with \( m \)th mode in \( x \)-direction and \( n \)th mode in \( y \)-direction.

Table 6
Comparison of non-dimensional natural frequencies \( \bar{\omega}_{mn} \) of simply-supported orthotropic rectangular plate

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>( \bar{\omega}_{mn} ) by various theories</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>EXACT (^a)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.0474</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.1033</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.1188</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.1694</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.1888</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.2180</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.2475</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.2624</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0.2969</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.3319</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.332</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.3476</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.3707</td>
</tr>
</tbody>
</table>

\(^a\) \( \bar{\omega}_{mn} = \omega_{mn} h \sqrt{\rho/\mu_{11}} \); \( h/a = 0.1 \), \( b/a = 1.0 \); (\( E_2/E_1 = 0.52500 \), \( G_{12}/E_1 = 0.26293 \), \( G_{13}/E_1 = 0.15991 \), \( G_{23}/E_1 = 0.26681 \), \( \mu_{12} = 0.44046 \), \( \mu_{21} = 0.23124 \)).

\(^b\) Taken from reference of Srinivas and Rao (1970).

\(^c\) Taken from reference of Reddy (1984).

Table 7
Comparison of non-dimensional natural frequencies \( \bar{\omega}_{mn} \) of simply-supported isotropic square plate

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>( \bar{\omega}_{mn} ) by various theories</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>EXACT (^b)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.0932</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.2226</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.3421</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.4171</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.5239</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>–</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.6889</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.7511</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>–</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>0.9268</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>–</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1.0889</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>–</td>
</tr>
</tbody>
</table>

\(^a\) Against an entry indicates that results/data are not available.

\(^b\) \( \bar{\omega}_{mn} = \omega_{mn} h \sqrt{\rho/G}; \ h/a = 0.1 \), \( b/a = 1.0 \); (\( E_2/E_1 = E \), \( G_{12} = G_{13} = G_{23} = G = [E/2(1+\mu)] \), \( \mu_{12} = \mu_{21} = \mu = 0.3 \)).

\(^b\) Taken from reference of Reddy and Phan (1985).
Using expressions (65) and (66) in the governing Eqs. (63) and (64), one obtains two equations, which can be written in matrix form, as follows:

\[
\begin{bmatrix}
K_{11mn} & 0 \\
0 & K_{22mn}
\end{bmatrix}
\begin{bmatrix}
W_{bmn} \\
W_{smn}
\end{bmatrix}
-
\omega_{mn}^2
\begin{bmatrix}
M_{11mn} & M_{12mn} \\
M_{12mn} & M_{22mn}
\end{bmatrix}
\begin{bmatrix}
W_{bmn} \\
W_{smn}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

(67)

where,

\[
M_{11mn} = \frac{\rho h^3}{12} \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] + \rho h
\]

\[
M_{12mn} = \rho h
\]

\[
M_{22mn} = \frac{\rho h^3}{1008} \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] + \rho h
\]

Eq. (67) is of standard eigen value form, and solving it one gets free vibration frequencies.

Here it is observed from the Eq. (67) that bending and shearing components are elastically uncoupled, but are inertially coupled.

The following nondimensionalized free vibration frequencies \( \bar{\omega}_{mn} \), \( \hat{\omega}_{mn} \) are tabulated in Tables 6 and 7.

\[
\bar{\omega}_{mn} = \omega_{mn} h \sqrt{\rho/Q_1}:	ext{ for orthotropic plate and}
\]

\[
\hat{\omega}_{mn} = \omega_{mn} h \sqrt{\rho/G}:	ext{ for isotropic plate.}
\]

Results are discussed separately after Table 7.

It should be noted that RPT can be adapted, apart from solving plates of rectangular planform, to solve plates of any other planform also. But, in such cases, analytical solutions would not be feasible. For solving such problems, it would be necessary to use some numerical techniques (e.g. devising and making use of finite element based on the present theory).

7. Discussion on results

Static analysis results using RPT are tabulated in Tables 1–5 and free vibration (predominantly bending frequency) results are tabulated in Tables 6 and 7. The tables also present solutions obtained using exact theory (Srinivas et al., 1970), Reddy’s theory [termed as ‘Higher-order Shear Deformation Plate Theory’ (HSDPT) in references of Reddy (1984), Reddy and Phan (1985)], Reissner’s theory [termed as ‘First-order Shear Deformation Plate Theory’ (FSDPT) in references of Reddy (1984), Reddy and Phan (1985)] and classical plate theory (CPT) taking into account rotary inertia [as reported in references of Reddy (1984), Reddy and Phan (1985)].

It is to be noted that results by Reddy (1984) are calculated for \( m,n = 1,3,\ldots,19 \). Whereas, present results are obtained using same values of \( m \) and \( n \) as those used for obtaining results using exact theory (Srinivas and Rao, 1970).

Exact theory results available in the literature (Srinivas et al., 1970; Srinivas and Rao, 1970) are used as basis for comparison of results obtained by various theories. The % error is calculated as follows:

\[
\% \text{ error} = \left( \frac{\text{value obtained by a theory}}{\text{corresponding value by exact theory}} - 1 \right) \times 100
\]

7.1. Discussion on static analysis results

- On results obtained for displacement
  - Orthotropic plate: Table 1 shows the comparison of static deflection at the center of simply-supported orthotropic plate obtained by various theories. RPT shows very good accuracy in results e.g., maximum error for the present results is 1.74% for thick square plate case \( (a/b = 1, h/a = 0.14) \). Whereas CPT has maximum error of –18.7% for \( a/b = 2, h/a = 0.14 \) case. Reddy’s and Reissner’s theories give marginally
improved results (maximum error in Reddy’s theory is 1.06%, maximum error in Reissner’s theory is −0.01%) as compared to those obtained by RPT.

- **Isotropic plate:** Table 2 shows the comparison of static deflection at the center of simply-supported isotropic plate obtained by various theories. CPT has −18.11% error for a thick rectangular plate (a/b = 2, h/a = 0.14). RPT shows very good accuracy in results e.g., 2.46% error for the same plate Reissner’s theory gives marginally improved results (−0.58% error for the same plate) in comparison with RPT. Reddy’s results are not available for displacement.

- **On results obtained for stresses for orthotropic plate**
  - Stress $\bar{\sigma}$: Table 3 shows comparison of stress $\bar{\sigma}$, obtained by various theories. CPT and Reissner’s theory give almost same stress results. RPT gives accurate stresses than CPT and Reissner’s theory for moderately thick plates (e.g., for $a/b = 0.5, h/a = 0.05, 0.1, 0.14$) but % error is more for thicker plate case (e.g., 5.67% error for rectangular plate with $a/b = 2, a/h = 0.14$, for the same case CPT and Reissner’s theory both have 3.5% error). Reddy’s theory shows good accuracy (i.e., maximum % error is about 0.63%).
  - Stress $\tau_s$: Table 4 shows comparison of stress $\tau_s$ obtained by various theories. Results using Reddy’s theory are not available. For a rectangular plate (having $a/b = 2.0, h/a = 0.14$) result for $\tau_s$ for RPT have maximum error of 2.99%, whereas CPT has −5.66% and Reissner’s theory has −1.97%, RPT results are more accurate than those of CPT. Reissner’s theory shows marginally improved accuracy than those of RPT.
  - Stress $\tau_x$: Table 5 shows comparison of shear stress $\tau_x$ obtained by various theories. Direct method of calculating shear stresses involves use of constitutive relations. Indirect method makes use of equilibrium equations. For example, in the case of CPT, transverse shear stresses cannot be obtained using shear stress and shear strain constitutive relations, and these are required to be obtained by indirect method. In this method, first stresses $\sigma_x, \sigma_y, \tau_{xy}$ are obtained. Then, these stresses are substituted in the equilibrium equations of the three dimensional theory of elasticity, and then integrating the equations and finding the constants of integrations, one obtains the expressions for transverse shear stresses $\tau_{xz}$ and $\tau_{yz}$. When the indirect method is used, results obtained by RPT are the best amongst the results presented. For a square plate case ($a/b = 1.0, a/h = 0.14$) results obtained by RPT have −3.92% error.

It is interesting to note that even though Reissner’s is a stress based approach, % error in shear stress results obtained using his approach is quiet high (e.g., 7.21% for rectangular plate having $a/b = 2.0, a/h = 0.14$). Using CPT, shear stresses can be found out only using indirect method and % error increases for thicker plates. Unlike CPT, using RPT shear stresses can be found out using direct method, but the % error is quite high (for thick plate, $a/b = 2.0, h/a = 0.14$ case error is 29.12%). Using Reddy’s theory, shear stresses obtained by direct as well as indirect methods show satisfactory results (around 5% error), but it is to be noted that stresses converge very slowly (Reddy, 1984) and results quoted for Reddy’s theory are only for $m = 1, 3, \ldots \ldots 19$. Use of more number of terms may lead to different conclusion.

### 7.2. Discussion on free vibration results

Results of free vibration frequencies for an orthotropic and also for isotropic square plate ($b/a = 1, h/a = 0.1$) are presented in Tables 6 and 7, respectively. Also, results quoted are in ascending order of frequencies. When $m = 1$ and $n = 1$, the frequency indicated is the fundamental frequency. The following observations can be made:

- **Orthotropic plate:** CPT results are not satisfactory for higher modes (e.g., error is 46%, when $m = 4, n = 2$). RPT gives good accuracy (e.g., maximum % error for the RPT results presented is 2.51% when $m = 3$ and $n = 3$). Reddy’s theory and Reissner’s theory give slightly better accuracy than RPT (e.g., for the results presented in respect of Reddy’s and Reissner’s theories, maximum % error is about 0.36%). It is to be noted that RPT has only one variable for capturing shear effects unlike other theories.
- **Isotropic plate:** It is observed from Table 7 that CPT results are not satisfactory for higher modes (e.g., error is 26%, when $m = 4, n = 4$). RPT gives good accuracy (e.g., error is −0.96%, when $m = 4, n = 4$). Reddy’s theory and Reissner’s theory give marginally accurate results than RPT (e.g., Reddy has −0.39% error and Reissner has −1.15% error, when $m = 4, n = 4$).
It should be noted that RPT involves only two unknown functions and two differential equations as against five unknown functions and five differential equations in case of Reddy’s theory, and three unknown functions and three differential equations in case of Reissner’s theory.

Moreover, in RPT, both the differential equations are only inertially coupled and there is no elastic coupling; and, therefore, the equations are easier to solve. Whereas, in Reddy’s theory as well as Reissner’s theory, all the differential equations are inertially as well as elastically coupled, and therefore, these equations are comparatively more difficult to solve.

Also unlike Reissner’s theory, RPT satisfies shear stress free boundary conditions, and also constitutive relations in respect of transverse shear stresses and strains (and, therefore, does not require shear correction factor).

CPT results are not satisfactory in respect of thick plates for static flexure as well as for free vibrations.

8. Concluding remarks

In this paper, a new two variable refined plate theory (RPT) is presented for orthotropic plate analysis. The efficacy of the theory has been demonstrated for static and free vibration problems.

The following points need to be noted in respect of RPT:

1. Use of the theory results in two fourth order governing differential equations. For static analysis case, both equations are fully uncoupled; whereas for dynamic analysis case, both equations are only inertially coupled and there is no elastic coupling at all. No other theory, to the best of the knowledge of the authors, has this feature.
2. Number of unknown functions involved in the theory is only two. Even in the Reissner’s and Mindlin’s theory (first order shear deformation theories), three unknown functions are involved.
3. The theory is variationally consistent.
4. The theory has strong similarity with the classical plate theory in many aspects (in respect of a governing equation, boundary conditions, moment expressions).
5. (a) The theory assumes displacements such that transverse shear stress variation is realistic (giving rise to shear stress free surfaces and parabolic variation of shear stress across the thickness). (b) Constitutive relations in respect of shear stresses and shear strains are satisfied (and, therefore, shear correction factor is not required).
6. The classical plate theory (CPT) comes out as a limiting case of RPT formulation (in situations wherein shear effects become ignorable, e.g., flexure of thin plate). Therefore, the finite elements based on RPT will be free from shear locking.
7. CPT involves use of only one unknown function and only one differential equation. Compared to CPT, use of RPT involves only one additional function and one additional number of differential equation. But, when RPT is used, gain in accuracy of results is not only substantial as compared to CPT, but almost comparable to higher order theories containing more number of unknown functions.
8. As mentioned in Section 4, RPT is equally applicable to thick as well as thin plate analysis. However, with increase in thickness of plate and increase in orthotropy of plate material, the results using RPT may show a trend (which is common to many other two-dimensional theories) of slight decrease in accuracy of results.

In conclusion, it can be said that for orthotropic plate analysis, RPT can be successfully utilized as it is the simplest yet accurate shear deformable theory with only two variables.

References


