Pseudo Generalized Youden Designs

Ashish Das¹, Daniel Horsley² and Rakhi Singh³

¹Indian Institute of Technology Bombay, Mumbai, India
²Monash University, Clayton, Australia
³IITB-Monash Research Academy, IIT Bombay, Mumbai, India

19th April 2017
Pseudo Generalized Youden Designs

Ashish Das

Indian Institute of Technology Bombay, Mumbai, India

Daniel Horsley

Monash University, Clayton, Australia

Rakhi Singh

IITB-Monash Research Academy, Mumbai, India

Abstract

Sixty years ago, Kiefer (1958) introduced generalized Youden designs (GYDs) for eliminating heterogeneity in two directions. A GYD is a row-column design whose \( k \) rows form a balanced block design (BBD) and whose \( b \) columns do likewise. Later Cheng (1981b) introduced pseudo Youden designs (PYDs) in which \( k = b \) and where the \( k \) rows and the \( b \) columns, considered together as blocks, form a BBD. Kiefer (1975b) proved a number of results on the optimality of GYDs. A PYD has the same optimality properties as a GYD. In the present paper, we introduce and investigate pseudo generalized Youden designs (PGYDs) which generalise both GYDs and PYDs. A PGYD is a row-column design where the \( k \) rows and \( b \) columns, considered together as blocks, form an equireplicate generalized binary variance balanced design. Every GYD is a PGYD and a PYD is exactly a PGYD with \( k = b \). We show, however, that there are situations where a PGYD exists but neither a GYD nor a PYD does. We obtain necessary conditions, in terms of \( v, k \) and \( b \), for the existence of a PGYD. Using these conditions, we provide an exhaustive list of parameter sets satisfying \( v \leq 25, k \leq 50, b \leq 50 \) for which a PGYD exists. We construct families of PGYDs using patchwork methods based on affine planes.

Keywords and phrases. generalized Youden design, pseudo Youden design, balanced block design, \( A \)-optimality, \( D \)-optimality, \( E \)-optimality
1 Introduction

Let $d$ be a row-column design for comparing $v$ treatments $\{1, \ldots, v\}$ via $kb$ experimental units arranged in $k$ rows and $b$ columns. In an additive and homoscedastic fixed effects two-way heterogeneity model, the information matrix ($C$-matrix) for estimating linear functions of the treatment effects using $d$ is given by

$$C_d = R_d - b^{-1} M_d M'_d - k^{-1} N_d N'_d + (kb)^{-1} r_d r'_d,$$

where

- $R_d = \text{diag}(r_1, r_2, \ldots, r_v)$ and $r_d = (r_1, r_2, \ldots, r_v)'$ where $r_i$ is the replication of the $i$th treatment for $i = 1, 2, \ldots, v$;
- $M_d = ((m_{i\ell}))$ is the $v \times k$ treatment-row incidence matrix where $m_{i\ell}$ counts the number of times treatment $i$ appears in $\ell$th row of $d$;
- $N_d = ((n_{i\ell}))$ is the $v \times b$ treatment-column incidence matrix where $n_{i\ell}$ counts the number of times treatment $i$ appears in $\ell$th column of $d$.

A design $d$ is said to be connected if it allows estimation of all treatment contrasts; equivalently, if $\text{rank}(C_d) = v - 1$. Let $D(v,k,b)$ denote the class of all connected row-column designs with $v$ treatments, $k$ rows and $b$ columns.

Sixty years ago, Kiefer (1958) introduced generalized Youden designs (GYDs) (though he called them generalized Youden squares, then) as a generalization to the usual latin squares and Youden squares. A GYD is a $k \times b$ row-column design which is balanced in both directions, that is, its rows form a balanced block design (BBD) and likewise for its columns. Kiefer (1975b) established optimality properties for GYDs. A BBD on $v$ treatments arranged in $b$ blocks each of size $k$ has the following properties:

(i) each treatment appears in each block $\lfloor k/v \rfloor$ times or $\lfloor k/v \rfloor + 1$ times, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$;
(ii) each treatment appears $kb/v$ times; and
(iii) $\sum_{\ell=1}^{b} n_{i\ell} n_{j\ell}$ is identical for all $1 \leq i < j \leq v$, where $n_{i\ell}$ is the number of appearances of treatment $i$ in block $\ell$.

When $k < v$, the BBD is the traditional balanced incomplete block design (BIBD) with parameters $(v, b, k)$. GYDs have been constructed primarily by Kiefer, Ruiz, Seiden, Wu and Ash. See for example, Kiefer (1975b), Ruiz and Seiden (1974), Seiden and Wu (1978), and Ash (1981).

Later Cheng (1981b) introduced pseudo Youden designs (PYDs) in which $k = b$ and where the $k$ rows and the $b$ columns, considered together as blocks, form a BBD. A PYD has the same optimality properties as a GYD. Constructions for PYDs have been
provided in Cheng (1981b), Cheng (1981a) and Ash (1981). Over the past thirty six years there have been no further innovations in GYDs and PYDs.

A matrix is said to be completely symmetric when all the diagonal elements are the same and all the off-diagonal elements are the same. Complete symmetry of the $C$-matrix is important in proving the optimality of a design. From (1.1), the $C$-matrix of a design $d$ can be written as

$$C_d = R_d - (kb)^{-1}(kM_dM_d' + bN_dN_d') + (kb)^{-1}r_d' r_d.$$

(1.2)

Extending Cheng’s observation on PYDs, it is clear that the $C$-matrix (1.2) has the same form that corresponds to a $k \times b$ GYD, so long as $[N_d : M_d]$ is the incidence matrix of an equireplicate generalized binary variance balanced block design, that is, $n_d$ is either $\lfloor k/v \rfloor$ or $\lfloor k/v \rfloor + 1$ for $\ell \in \{1, \ldots, b\}$, $m_d$ is either $\lfloor b/v \rfloor$ or $\lfloor b/v \rfloor + 1$ for $\ell \in \{1, \ldots, k\}$, and $kM_dM_d' + bN_dN_d'$ is completely symmetric. We shall call such a row-column design a pseudo generalized Youden design (PGYD). Clearly, when $k = b$, a PGYD is a PYD and thus a PGYD is a generalization of Cheng’s notion of a PYD for situations when $k \neq b$. Also, every GYD is a PGYD. We will show, however, that there are situations where a PGYD exists but neither a GYD nor a PYD does. Furthermore, using the technique of Kiefer (1975b) one can show that among all designs in $D(v, k, b)$, a PGYD is $A$- and $E$-optimal, and is $D$-optimal if $v \neq 4$.

In Section 2, we obtain necessary conditions, in terms of $v$, $k$ and $b$, for the existence of a PGYD. Using these conditions, in Section 3 we provide an exhaustive list of parameter sets satisfying $v \leq 25, k \leq 50, b \leq 50$. In each case we establish that a PGYD exists, except for one where we indicate non-existence. In Section 3, we also construct families of PGYDs using patchwork methods based on affine planes.

## 2 Necessary conditions for existence of PGYDs

A row-column design $d$ with $v$ treatments, $k$ rows and $b$ columns is called a two-way generalized binary design if each treatment occurs $\lfloor k/v \rfloor$ or $\lfloor k/v \rfloor + 1$ times in each of the $b$ columns and $\lfloor b/v \rfloor$ or $\lfloor b/v \rfloor + 1$ times in each of the $k$ rows.

A two-way generalized binary design $d$ with $v$ treatments, $k$ rows and $b$ columns is a PGYD if the following conditions hold:

(i) Each treatment occurs exactly $r = kb/v$ times in $d$, and

(ii) $kM_dM_d' + bN_dN_d'$ is completely symmetric.
When (i) holds we say that the design is equireplicate. It is easy to see that a two-way generalized binary design $d$ with $C_d$ completely symmetric is necessarily equireplicate.

**Definition 1.** Let $V$ be a set of $v$ treatments. Let $d$ be a two-way generalized binary design on $V$ with $k$ rows and $b$ columns. In what follows, let $k = k'v + k''$ and $b = b'v + b''$ where $k' = \lfloor k/v \rfloor$, $b' = \lfloor b/v \rfloor$ and $k''$ and $b''$ are non-negative integers. For $u \in \{1, \ldots, k\}$ let $R_u$ be the set of treatments that occur $b'+1$ times in row $u$ and for $w \in \{1, \ldots, b\}$ let $C_w$ be the set of treatments that occur $k'+1$ times in column $w$. For any two treatments $i, j$ in $V$, we define

$\delta_{ij} = |\{u : \{i, j\} \subseteq R_u\}|; \text{ and } \lambda_{ij} = |\{w : \{i, j\} \subseteq C_w\}|.$

Let the collection $\{R_1, \ldots, R_k\}$ be denoted by $d_R$ and the collection $\{C_1, \ldots, C_b\}$ be denoted by $d_C$.

**Theorem 2.** Let $d$ be an equireplicate two-way generalized binary design with $v \geq 2$ treatments, $k$ rows and $b$ columns. Then $d$ is a PGYD if and only if $k\delta_{ij} + b\lambda_{ij}$ is identical for any two distinct treatments $i$ and $j$ in $V$.

**Proof.** We first consider the diagonal entries of $kM_dM'_d + bN_dN'_d$. Because each treatment occurs $b' + 1$ times in exactly $r - kb'$ rows and exactly $b'$ times in the rest, it can be seen that the diagonal elements of $M_dM'_d$ are all equal. Similarly, each treatment occurs $k' + 1$ times in exactly $r - bk'$ columns and exactly $k'$ times in the rest, and the diagonal elements of $N_dN'_d$ are all equal. Thus the diagonal elements of $kM_dM'_d + bN_dN'_d$ are all equal.

We now consider the off-diagonal entries of $kM_dM'_d + bN_dN'_d$. Let $i$ and $j$ be distinct treatments in $V$ and let $\nu_z = |\{u : |\{i, j\} \cap R_u| = z\}|$ for $z \in \{0, 1, 2\}$. The $(i, j)$ entry in $M_dM'_d$ is

$$(b')^2\nu_0 + b'(b' + 1)\nu_1 + (b' + 1)^2\nu_2 = k(b')^2 + b'\nu_1 + (2b' + 1)\nu_2$$

where the equality follows because $\nu_0 = k - \nu_1 - \nu_2$. Because there are exactly $r - kb'$ rows in which $i$ occurs $b' + 1$ times and $r - kb'$ rows in which $j$ occurs $b' + 1$ times, $\nu_1 = 2(r - kb' - \nu_2)$. Also, $\nu_2 = \delta_{ij}$. Thus, the $(i, j)$ entry in $M_dM'_d$ is

$$2b'r - k(b')^2 + \delta_{ij}.$$ 

Similarly, it can be established that the $(i, j)$ entry in $N_dN'_d$ is

$$2k'r - b(k')^2 + \lambda_{ij}.$$ 

4
Thus it can be seen that the off-diagonal elements of $kM_dM_d’ + bN_dN_d’$ are all equal if and only if $k\delta_{ij} + b\lambda_{ij}$ is identical for any two treatments $i$ and $j$ of $V$.  

Our next result provides necessary conditions for the existence of a PGYD.

**Theorem 3.** Necessary conditions for $d$ in $D(v, k, b)$ to be a PGYD are:

1. $k + b \geq v$.
2. $\frac{k(r-kb')(b''-1)+b(r-kb')(k''-1)}{v-1} = t$ is an integer.
3. There exist $p \geq 1$ pairs of non-negative integers $(m_1, n_1), \ldots, (m_p, n_p)$ such that, for $\ell = 1, \ldots, p$,
   - $(i)$ $km_\ell + bn_\ell = t$,
   - $(ii)$ $2r - 2k' - k \leq m_\ell \leq r - k'$ and $2r - 2k' - b \leq n_\ell \leq r - b'$.
4. There exist non-negative integers $z_1, \ldots, z_p$ such that
   - $(i)$ $\sum_{\ell=1}^{p} z_\ell = \binom{v}{2}$,
   - $(ii)$ $\sum_{\ell=1}^{p} z_{m_\ell} = k \binom{b''}{2}$,
   - $(iii)$ $\sum_{\ell=1}^{p} z_{n_\ell} = b \binom{k''}{2}$.

**Proof.** We provide the proofs for each of the conditions (1) – (4) below.

Condition (1): Corresponding to a PGYD, there exists a variance balanced block design with a $v \times (k + b)$ incidence matrix $[N_d : M_d]$. Thus following Dey (1975), $k + b \geq v$.

Condition (2): This follows from the requirement that $k\delta_{ij} + b\lambda_{ij}$ in Theorem 2 is an integer. Let $t = k\delta_{ij} + b\lambda_{ij}$. To find its value, we note that the total number of pairs of treatments in blocks of $d_R$ and $d_C$ are respectively,

$$\sum_{i<j} \delta_{ij} = k \binom{b''}{2} \quad \text{and} \quad \sum_{i<j} \lambda_{ij} = b \binom{k''}{2}.$$  

This is so because in $d_R$ there are $k$ blocks and each block is of size $b''$, and in $d_C$ there are $b$ blocks and each block is of size $k''$. Therefore, summing over all $\binom{v}{2}$ treatment pairs, we get $\sum_{i<j}(k\delta_{ij} + b\lambda_{ij}) = \sum_{i<j} t$, which using (2.1) gives

$$t = \frac{k^2 \binom{b''}{2} + b^2 \binom{k''}{2}}{\binom{v}{2}} = \frac{k(r-kb')(b''-1)+b(r-kb')(k''-1)}{v-1}.$$  

Condition (3): Index the distinct pairs in $\{(\delta_{ij}, \lambda_{ij}) : 1 \leq i < j \leq v\}$ as $(m_1, n_1), \ldots, (m_p, n_p)$. Using Condition (2) above, $km_\ell + bn_\ell = t$ for $\ell = 1, \ldots, p$. From the proof of Theorem 2, since $\nu_2 = \delta_{ij}$, $\nu_1 = 2(r-kb' - \nu_2) = 2(r-kb' - \delta_{ij})$ and $\nu_0 = k - \nu_1 - \nu_2 = k - 2r + 2kb' + \delta_{ij}$
must be non-negative, we have \(2r - 2kb - k \leq m_\ell \leq r - kb\) for \(\ell = 1, \ldots, p\). Similarly, we have \(2r - 2bk - b \leq n_\ell \leq r - bk\) for \(\ell = 1, \ldots, p\).

Condition (4): Let \(z_\ell = |\{(i, j) : (\delta_{ij}, \lambda_{ij}) = (m_\ell, n_\ell), 1 \leq i < j \leq v\}|, \ell = 1, \ldots, p\). It is clear that \(\sum_{\ell=1}^{p} z_\ell = \binom{v}{2}\). Also, from (2.1),

\[
\sum_{\ell=1}^{p} z_\ell m_\ell = k\binom{b''}{2} \quad \text{and} \quad \sum_{\ell=1}^{p} z_\ell n_\ell = b\binom{k''}{2}.
\]

A \(k \times b\) row-column setting with \(v\) treatments is called regular if \(k \equiv 0 \pmod{v}\) or \(b \equiv 0 \pmod{v}\). Accordingly, a PGYD with parameters \(v, k = k'v + k'', b = b'v + b''\) is regular if \(k'' = 0\) or \(b'' = 0\); otherwise it is said to be non-regular. For a GYD with parameters \(v, k, b\), there exists two corresponding BIBDs with parameters \((v, k', b')\) and \((v, b, k'')\). A regular PGYD reduces to a regular GYD, existence of which depends solely on the existence of a corresponding BIBD. Thus, we restrict ourselves to non-regular PGYDs for which \(v\) divides neither \(k\) nor \(b\) (that is, \(k'' \neq 0\) and \(b'' \neq 0\)). Also, without loss of generality, we can consider \(k \leq b\). We give two results which follow from Theorem 3.

**Corollary 4.** Necessary conditions for the existence of a non-GYD PGYD, in addition to necessary conditions (1) and (2) in Theorem 3 are,

- \((3')\) \(p \geq 2\), in the condition (3), and
- \((4')\) at least two of the \(z_\ell\)’s are non-zero, in the condition (4).

**Corollary 5.** Necessary conditions for the existence of a non-regular GYD are,

- \((1')\) \(k \geq v\) and \(b \geq v\), and
- \((2')\) \(k\binom{b''}{2}/\binom{v}{2}\) and \(b\binom{k''}{2}/\binom{v}{2}\) are integers.

For a GYD two variance balanced block designs with incidence matrices \(N_d\) and \(M_d\) should exist. Thus, \((1')\) follows from Dey (1975). Also, \((2')\) follows directly from the condition (4) of Theorem 3 since for a GYD exactly one of the \(z_\ell\)’s should be non-zero.

**Remark 6.** In addition to the necessary conditions for a non-regular GYD as given in Corollary 5, additional parametric conditions for the existence of the corresponding BIBDs, as given in Theorem 10.3.1 and Thoerem 16.1.3 of Hall (1998), must also hold.

Ash (1981) gave constructions of GYDs for all parameter sets satisfying \(v \leq 25, k \leq 50, b \leq 50\) except \(v = 25, k = b = 40\), for which a PYD was provided. Covering the same
parametric range, in Section 3 we provide an exhaustive list of admissible parameter sets of non-regular PGYDs for \( v \leq 25, k \leq b \leq 50 \) and indicate a construction for each, except for one which is non-existent. A few families of PGYDs are constructed in Section 3. Several new PGYDs, which are neither GYDs nor PYDs, are constructed. Extending the parametric range to \( 26 \leq v \leq 50, k \leq b \leq 50 \), there are only three additional parameter sets \((v, k, b) = (28, 18, 42), (36, 42, 42), (49, 28, 28)\) for which a non-regular PGYD is possible. Incidentally, a GYD is non-existent for each of the three parameter sets. For \((49, 28, 28)\), though a GYD is non-existent since Condition (1'\) of Corollary 5 is violated, there exists a PYD constructed by Cheng (1981a). For \((36, 42, 42)\), the non-existence of a GYD follows from Remark 6. Finally, for \((28, 18, 42)\), a GYD is non-existent since Condition (1'\) of Corollary 5 is violated. Whether there exists a non-GYD PGYD for the parameter sets \((28, 18, 42)\) or \((36, 42, 42)\) is currently unknown.

3 Construction of PGYDs

The constructions presented in this section are patchwork methods which go back to Kiefer (1975a). These constructions rely heavily on affine planes. For our purposes an affine plane of order \( q \) is a BIBD \((V, B)\) where \( V \) is a set of \( q^2 \) treatments, \( B \) is a set of \( q(q+1) \) blocks of size \( q \), and any two treatments appear together in exactly one block. The blocks of such a design can be partitioned into \( q+1 \) parallel classes each containing \( q \) blocks such that any two blocks from the same parallel class are disjoint and any two blocks from different parallel classes intersect in exactly one point. We will use this property frequently. An affine plane of order \( q \) is known to exist whenever \( q \) is a prime power. We will also sometimes consider complements of affine planes. For a block \( B \) of an affine plane on treatment set \( V \), let \( B^c = V \setminus B \) and for a parallel class \( P \) of such a plane, let \( P^c = \{B^c : B \in P\} \).

**Lemma 7.** Let \( m, n \) and \( v \) be positive integers with \( n \equiv 0 \) (mod \( v \)), let \( V \) be a set of \( v \) treatments, and let \( S_1, \ldots, S_n \) be \( m \)-subsets of \( V \). If every treatment occurs exactly \( mn/v \) times in the collection \( \{S_1, \ldots, S_n\} \), then there is an \( m \times n \) matrix \( A \) such that the set of treatments in the \( w \)th column of \( A \) is \( S_w \) and each treatment in \( V \) occurs \( n/v \) times in each row of \( A \).

**Proof.** Let \( G \) be the bipartite graph with parts \( \{c_1, \ldots, c_n\} \) and \( V \) such that the set of vertices adjacent to \( c_w \) is \( S_w \) for \( w \in \{1, \ldots, n\} \). Then \( \deg_G(c_w) = m \) for \( w \in \{1, \ldots, n\} \) and, by our hypothesis, \( \deg_G(i) = mn/v \) for each \( i \in V \). By a result of de Werra (1971)
the edges of $G$ can be colored with $m$ colours, say $1, \ldots, m$, such that each vertex in
\{c_1, \ldots, c_n\} is incident with exactly one edge of each color, and each vertex in $V$ is
incident with exactly $n/v$ edges of each color.

Form $A$ by placing in the $(u, w)$ position the unique element $i$ of $V$ such that the edge
c_{wi} of $G$ is assigned color $u$. That the set of treatments in the $w$th column of $A$ is $S_w$
follows from the definition of $G$. That each treatment in $V$ occurs $n/v$ times in each row
of $A$ follows from the fact that each vertex in $V$ is incident with exactly $n/v$ edges of
each color. \hfill \square

Lemma 8. Let $(V, B)$ be an affine plane of order $q$, and let $P_1, \ldots, P_{q-1}$ and $Q_1, \ldots, Q_{q-1}$
be parallel classes of $(V, B)$ (not necessarily distinct) such that $P_x \neq Q_y$ for $x, y \in
\{1, \ldots, q-1\}$.

(i) For any $x, y \in \{1, \ldots, q-1\}$ there is a $q \times q$ matrix $A$ such that the sets of treatments
in the rows of $A$ are the elements of $P_x$ and the sets of treatments in the columns
of $A$ are the elements of $Q_y$.

(ii) For any $x \in \{1, \ldots, q-1\}$ there is $q \times (q^2 - q)$ matrix $A$ such that the sets of treatments
in the rows of $A$ are the elements of $P_x^c$ and the sets of treatments in the columns
of $A$ are the elements of $Q_1, \ldots, Q_{q-1}$.

(iii) There is a $(q^2 - q) \times (q^2 - q)$ matrix $A$ such that the sets of treatments in the rows
of $A$ are the elements of $P_1^c, \ldots, P_{q-1}^c$ and the sets of treatments in the columns
of $A$ are the elements of $Q_1^c, \ldots, Q_{q-1}^c$.

Proof. For $x, y \in \{1, \ldots, q-1\}$, let $P_x = \{P_{x,1}, \ldots, P_{x,q}\}$ and let $Q_y = \{Q_{y,1}, \ldots, Q_{y,q}\}$.

Case (i): We will show that there exists a $q \times q$ matrix $A$ such that the set of treatments
in the $u$th row of $A$ is $P_{x,u}$ and the set of treatments in the $w$th column of $A$ is $Q_{y,w}$.
Because $x \neq y$, $|P_{x,u} \cap Q_{y,w}| = 1$ for all $u, w \in \{1, \ldots, q\}$. So $A$ can be obtained by
placing the unique element of $P_{x,u} \cap Q_{y,w}$ in the $(u, w)$ position.

Case (ii): As in the proof of (i) there is, for each $y \in \{1, \ldots, q-1\}$, a $q \times q$ matrix $A_y$
such that the set of treatments in the $u$th row of $A_y$ is $P_{x,u+y}$ and the set of treatments
in the $w$th column of $A_y$ is $Q_{y,w}$ (where the subscripts are considered modulo $q$). We take

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_{q-1} \end{bmatrix}.$$ 

The set of treatments in the $u$th row of $A$ is $P_{x,u}$.

Case (iii): As in the proof of (i) there is, for each $x, y \in \{1, \ldots, q-1\}$, a $q \times q$
matrix $A_{x,y}$ such that the set of treatments in the $u$th row of $A_{x,y}$ is $P_{x,u+y}$ and the set of
treatments in the $w$th column of $A_{x,y}$ is $Q_{y,w+x}$ (where the subscripts are considered
modulo $q$). We take

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,q-1} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,q-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1,1} & A_{q-1,2} & \cdots & A_{q-1,q-1} \end{bmatrix}.$$  

The set of treatments in the $u$th row of $A$ is $P_{x,u'}^c$ where $u = (x - 1)q + u'$ and $u' \in \{1, \ldots, q\}$. Similarly the set of treatments in the $w$th column of $A$ is $Q_{y,w'}^c$ where $w = (y - 1)q + w'$ and $w' \in \{1, \ldots, q\}$.

The following Theorem gives us four families of PGYDs based on the residues of $k$ and $b$ modulo $q^2$. Let $\gcd(b, k)$ denote the greatest common divisor of $b$ and $k$.

**Theorem 9.** Let $q$ be a prime power. There exists a PGYD with $v = q^2$ treatments, $k$ rows and $b$ columns if

(i) $k \equiv \pm q \pmod{q^2}$;

(ii) $b \equiv \pm q \pmod{q^2}$;

(iii) $k = k'q(q + 1) + \frac{b}{\gcd(b, k)}q(q + 1 - n)$ and $b = b'q(q + 1) + \frac{k}{\gcd(b, k)}nq$ for some $n \in \{0, \ldots, q\}$ and non-negative integers $k'$ and $b'$.

**Proof.** From (i) and (ii) we have $k = k'q^2 + k''$ and $b = b'q^2 + b''$ where $k'', b'' \in \{q, q^2 - q\}$ and $k'$ and $b'$ are non-negative integers. Let $(V, B)$ be an affine plane order $q$ and let its parallel classes be $P_1, \ldots, P_{q+1}$. Let $g = \gcd(b, k)$.

Let $x_1, \ldots, x_{k/q}$ be the unique non-decreasing sequence of indices from $\{1, \ldots, q + 1\}$ such that each index in $\{1, 2, \ldots, n\}$ occurs $k^*$ times in the sequence and each index in $\{n + 1, \ldots, q + 1\}$ occurs $k^* + b/g$ times in the sequence. For $u \in \{1, \ldots, k/q\}$, let $R_u = P_{x,u}$ if $b'' = q$ and $R_u = P_{x,u}^c$ if $b'' = q^2 - q$. Let $y_1, \ldots, y_{b/q}$ be the unique non-increasing sequence of indices from $\{1, 2, \ldots, n\}$ such that each index in $\{1, 2, \ldots, n\}$ occurs $b^* + k/g$ times in the sequence and each index in $\{n + 1, \ldots, q + 1\}$ occurs $b^*$ times in the sequence. For $w \in \{1, \ldots, b/q\}$, let $C_w = P_{y,w}$ if $k'' = q$ and $C_w = P_{y,w}^c$ if $k'' = q^2 - q$.

We will form the required design as

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

where

- $W$ is a $(k - k'') \times (b - b'')$ matrix such that each treatment occurs the same number of times in each row of $W$ and the same number of times in each column of $W$.  

9
• $X$ is a $(k-k'') \times b''$ matrix such that the sets of treatments in the rows of $X$ are the elements of $\mathcal{R}_1, \ldots, \mathcal{R}_{(k-k'')/q}$ and each treatment occurs $k'$ times in each column of $X$.

• $Y$ is a $k'' \times (b - b'')$ matrix such that the sets of treatments in the columns of $Y$ are the elements of $\mathcal{C}_1, \ldots, \mathcal{C}_{(b - b'')/q}$ and each treatment occurs $b'$ times in each row of $Y$.

• $Z$ is a $k'' \times b''$ matrix such that the sets of treatments in the rows of $Z$ are the elements of $\mathcal{R}_{(b - b'')/q+1}, \ldots, \mathcal{R}_{b/q}$ and the sets of treatments in the columns of $Z$ are the elements of $\mathcal{C}_{(k-k'')/q+1}, \ldots, \mathcal{C}_{k/q}$.

We will first show that such a design is a PGYD and then show that we can construct matrices $W$, $X$, $Y$ and $Z$ with the required properties.

It is clear that such a design is an equireplicate two-way generalized binary design. So to show the design is a PGYD it suffices, by Theorem 2, to show that $k\delta_{ij} + b\lambda_{ij}$ is identical for each pair of distinct treatments $(i, j)$. Let $(i, j)$ be a pair of distinct treatments. Define

$$\gamma_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ occur together in a block in } P_1 \cup \cdots \cup P_n; \\ 0, & \text{if } i \text{ and } j \text{ occur together in a block in } P_{n+1} \cup \cdots \cup P_{q+1}. \end{cases}$$

Note that $i$ and $j$ occur together in $1 - \gamma_{ij}$ blocks in $P_{n+1} \cup \cdots \cup P_{q+1}$. Note also that $i$ and $j$ occur together in $q - 1$ blocks of $P_{q+1}$ if $i$ and $j$ occur together in a block of $P_{q}$ and $i$ and $j$ occur together in $q - 2$ blocks of $P_{q}$ otherwise. Then from our construction we can calculate that $\lambda_{ij}$ and $\delta_{ij}$ are as given below.

$$\lambda_{ij} = \begin{cases} b^* + \frac{k}{g} \gamma_{ij}, & \text{if } k'' = q; \\ b^*(q^2 - q - 1) + \frac{k}{g}(n(q - 2) + \gamma_{ij}), & \text{if } k'' = q^2 - q. \end{cases}$$

$$\delta_{ij} = \begin{cases} k^* + \frac{b}{g}(1 - \gamma_{ij}), & \text{if } b'' = q; \\ k^*(q^2 - q - 1) + \frac{b}{g}((q + 1 - n)(q - 2) + (1 - \gamma_{ij})), & \text{if } b'' = q^2 - q. \end{cases}$$

Considering four cases according to the values of $k''$ and $b''$, it is easy to check that the value of $k\delta_{ij} + b\lambda_{ij}$ is independent of $\gamma_{ij}$. Hence $k\delta_{ij} + b\lambda_{ij}$ is identical for each pair of distinct treatments $(i, j)$ and the design is a PGYD.

We now show that we can construct matrices $W$, $X$, $Y$ and $Z$ with the required properties. It is easy to form $W$ by tiling $q^2 \times q^2$ latin squares. Because each treatment appears once in each parallel class and appears $q-2$ times in the complement of each parallel class, and because $b - b'' \equiv 0 \pmod{q^2}$, Lemma 7 can be used to construct a matrix $Y$ with the required properties. Similarly, by applying Lemma 7 and taking a transpose,
a matrix $X$ with the required properties can be constructed. Finally, Lemma 8 yields a matrix $Z$ with the required properties provided that the sets $\{x_{(k-k''\prime)/q+1}, \ldots, x_{k/q}\}$ and $\{y_{(b-b''\prime)/q+1}, \ldots, y_{b/q}\}$ are disjoint. We complete the proof by establishing this claim.

When $k'' = b'' = q$, $\{y_{b/q}\} = \{1\}$ and $\{x_{k/q}\} = \{q + 1\}$. When $b'' = q$ and $k'' = q^2 - q$, $\{y_{b/q}\} = \{1\}$ and $\{x_{(k-k'\prime)/q+1}, \ldots, x_{k/q}\} \subseteq \{3, \ldots, q + 1\}$. When $b'' = q^2 - q$ and $k'' = q$, $\{x_{k/q}\} = \{q + 1\}$ and $\{y_{(b-b''\prime)/q+1}, \ldots, y_{b/q}\} \subseteq \{1, \ldots, q - 1\}$. In each of these cases the claim is true, so we may assume that $b'' = k'' = q^2 - q$ and $q \neq 2$. We consider two cases according to whether $b = k$.

Suppose first that $k \neq b$. We are assuming $k \leq b$ without loss of generality, so $k < b$. Then

$$g = q \gcd(b'q + q - 1, k'q + q - 1) = q \gcd(b'q + q - 1, (b' - k')q) \leq q(b' - k') \leq b'q,$$

where the last equality follows because $(b' - k')q = (b'q + q - 1) - (k'q + q - 1)$ and the first inequality follows because $\gcd(b'q + q - 1, 1) = 1$. So we have $b'/g \geq (b'q^2 + q^2 - q)/(b'q) > q$. Thus $\{x_{(k-k''\prime)/q+1}, \ldots, x_{k/q}\} = \{q + 1\}$. Obviously $\{y_{(b-b''\prime)/q+1}, \ldots, y_{b/q}\} \subseteq \{1, \ldots, q - 1\}$, and the claim follows.

Now suppose that $k = b$. Then it follows from (iii) that $k = b = b^*q(q + 1) + nq$ where $n \in \{0, (q + 1)/2\}$. So, because $b \equiv -q \pmod{q^2}$, $b^* \equiv q - n - 1 \pmod{q}$. Thus, it must be the case that $n = 0$ and $b^* \geq 2$ or that $n = (q + 1)/2$ and $b^* \geq 1$ or that $n = (q + 1)/2$, $b^* = 0$ and $q = 3$. In each of these cases it can be verified that $\{y_{(b-b''\prime)/q+1}, \ldots, y_{b/q}\} \subseteq \{1, \ldots, [(q + 1)/2]\}$ and $\{x_{(k-k''\prime)/q+1}, \ldots, x_{k/q}\} \subseteq \{[(q + 3)/2], \ldots, q + 1\}$ where $[z]$ denotes the smallest integer greater than or equal to $z$.

Theorem 9 produces a PYD when $k = b$. In this case it must be that $n = 0$ or $n = (q + 1)/2$. The construction in Theorem 2.2 of Cheng (1981b) necessarily requires that $b \equiv q \pmod{q^2}$ and produces designs for parameter sets covered by Theorem 9. However, Theorem 9 also produces PDs for parameter sets not covered by Cheng’s construction. In particular, it does so when $b \equiv -q \pmod{q^2}$, as stated in the following corollary.

**Corollary 10.** Let $q$ be a prime power and $a$ be a positive integer. Then a PYD with $v = q^2$, $k = b = aq^2 - q$ exists (i) when $q$ is odd and $a \equiv -1 \pmod{q}$ and $q$ is even and $a \equiv -1 \pmod{q + 1}$.

**Remark 11.** Theorem 9 gives a GYD when $n = 0$ and Corollary 10 gives a GYD when $a \equiv -1 \pmod{q + 1}$.
**Theorem 12.** Let $d$ be a GYD with $k$ rows, $k$ columns and $v$ treatments, of the form
\[
\begin{bmatrix}
W & X \\
Y & Z
\end{bmatrix}
\]
where $W$ is $vk' \times vk'$ and is formed by tiling latin squares of order $v$, $Z$ is $k'' \times k''$, and each treatment occurs $k'$ times in each column of $X$ and $k'$ times in each row of $Y$. If there is a pair of treatments that occur together in the columns of $Z$ a different number of times from in the rows of $Z$, then the design $d^*$ formed from $d$ by replacing $Z$ with $Z^T$ is a PYD that is not a GYD.

**Proof.** Using the logic of Theorem 2, it follows that in the GYD $d$ each pair of distinct treatments $(i,j)$ appears $\mu$ times in the rows of $[X^T : Z^T]^T$ and $\mu$ times in the columns of $[Y : Z]$ for some positive integer $\mu$.

Let $(i,j)$ be a pair of distinct treatments. Say $(i,j)$ appears $\delta_{ij}^Z$ times in the rows of $Z$ and $\lambda_{ij}^Z$ times in the columns of $Z$. So $(i,j)$ appears $(\mu - \delta_{ij}^Z)$ times in the rows of $X$ and $(\mu - \lambda_{ij}^Z)$ times in the columns of $Y$. Then $d^*$ has the form
\[
\begin{bmatrix}
W & X \\
Y & Z^T
\end{bmatrix}
\]
and $(i,j)$ appears $\mu - \delta_{ij}^Z + \lambda_{ij}^Z$ times in the rows of $[X^T : Z]^T$ and $\mu - \lambda_{ij}^Z + \delta_{ij}^Z$ times in the columns of $[Y : Z^T]$. So, in $d^*$, $k\lambda_{ij} + k\delta_{ij} = k(\mu - \lambda_{ij}^Z + \delta_{ij}^Z + \mu - \delta_{ij}^Z + \lambda_{ij}^Z) = 2k\mu$.

Thus, in $d^*$, $k\lambda_{ij} + k\delta_{ij}$ is identical for any pair $(i,j)$ of distinct treatments, and $d^*$ is a PYD by Theorem 2. However, by our hypotheses, there is some pair $(i,j)$ of distinct treatments that appears $\delta_{ij}^Z$ times in the rows of $Z$ and $\lambda_{ij}^Z$ times in the columns of $Z$ where $\delta_{ij}^Z \neq \lambda_{ij}^Z$. So, using our arguments above, $\lambda_{ij} \neq \delta_{ij}$ in $d^*$ because $\mu - \delta_{ij}^Z + \lambda_{ij}^Z \neq \mu - \lambda_{ij}^Z + \delta_{ij}^Z$. Therefore, $d^*$ is not a GYD. 

**Remark 13.** Any GYD with $k = b$ constructed according to the proof of Theorem 9 will satisfy the conditions of Theorem 12. To see this, note that in the proof of Theorem 9, $1 \in \{y(k-k'')/q+1, \ldots, yk/q\}$ but $1 \notin \{x(b-b'')/q+1, \ldots, xb/q\}$. It follows that any pair of treatments that appears in a block in $P_1$ will appear more often in the columns of $Z$ than in the rows of $Z$.

Table 1 and Table 2 together cover all the non-regular PGYDs in the parameteric range $v \leq 25, k \leq b \leq 50$ satisfying Theorem 3. Table 1 gives parametric list of all possible non-regular GYDs satisfying the necessary conditions of Corollary 5. The constructions for the GYDs have been provided by Ash (1981) except when $v = 25, k = b = 40$, for
which a GYD construction is currently not known and an exclamation (!) has been marked against $v = 25$ to indicate the same. However, Ash (1981) provided a non-GYD PYD. Also, following Remark 6, for $v = 15, k = 21, b = 35$ an asterisk (*) has been marked against $v = 15$ to indicate non-existence of the GYD. Using Corollary 4 and Theorem 12, the table also exhibits the non-existence of, or constructions for, non-GYD PGYDs. The parameters in Table 1 where a non-GYD PGYD exists can be obtained from Theorem 9 and Theorem 12 (see Remark 13), except for $(8, 14, 28), (8, 28, 28), (9, 24, 48)$ and $(10, 36, 45)$. However, a non-GYD PGYD $(8, 28, 28)$ can be obtained by applying Theorem 12 to the GYD $(8, 28, 28)$ provided in Ash (1981) and a non-GYD PGYD $(9, 24, 48)$ is provided in the Appendix.

In Table 2, following Corollary 4 and Corollary 5, we list the non-regular PGYD parameter sets where GYDs are non-existent. Other than two designs for $(8, 20, 50)$ and $(18, 12, 48)$ provided in the Appendix, the constructions for the non-GYD PGYDs follow from Theorem 9.

Table 1: Non-GYD PGYD solutions or non-existence where GYDs exist for $v \leq 25, k \leq b \leq 50$

<table>
<thead>
<tr>
<th>$v$</th>
<th>$k$</th>
<th>$b$</th>
<th>Non-GYD PGYD?</th>
<th>$v$</th>
<th>$k$</th>
<th>$b$</th>
<th>Non-GYD PGYD?</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6</td>
<td>6</td>
<td>Yes: Theorem 12</td>
<td>8</td>
<td>28</td>
<td>28</td>
<td>Yes: Theorem 12</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>18</td>
<td>No: Corollary 4 (3')</td>
<td>8</td>
<td>28</td>
<td>42</td>
<td>No: Corollary 4 (4')</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>30</td>
<td>No: Corollary 4 (3')</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>Yes: Theorem 12</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>42</td>
<td>No: Corollary 4 (3')</td>
<td>9</td>
<td>12</td>
<td>24</td>
<td>No: Corollary 4 (4')</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>18</td>
<td>Yes: Theorem 12</td>
<td>9</td>
<td>12</td>
<td>48</td>
<td>No: Corollary 4 (3')</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>30</td>
<td>No: Corollary 4 (4')</td>
<td>9</td>
<td>24</td>
<td>24</td>
<td>Yes: Theorem 12</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>42</td>
<td>No: Corollary 4 (3')</td>
<td>9</td>
<td>24</td>
<td>48</td>
<td>Yes: Appendix</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>30</td>
<td>Yes: Theorem 12</td>
<td>9</td>
<td>48</td>
<td>48</td>
<td>Yes: Theorem 12</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>42</td>
<td>No: Corollary 4 (4')</td>
<td>10</td>
<td>15</td>
<td>36</td>
<td>No: Corollary 4 (3')</td>
</tr>
<tr>
<td>4</td>
<td>42</td>
<td>42</td>
<td>Yes: Theorem 12</td>
<td>10</td>
<td>18</td>
<td>45</td>
<td>No: Corollary 4 (3')</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>15</td>
<td>No: Corollary 4 (3')</td>
<td>10</td>
<td>36</td>
<td>45</td>
<td>Unknown</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>45</td>
<td>No: Corollary 4 (3')</td>
<td>12</td>
<td>33</td>
<td>44</td>
<td>No: Corollary 4 (3')</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>20</td>
<td>No: Corollary 4 (4')</td>
<td>15</td>
<td>21</td>
<td>35</td>
<td>No: Corollary 4 (4')</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>40</td>
<td>No: Corollary 4 (3')</td>
<td>15</td>
<td>35</td>
<td>42</td>
<td>No: Corollary 4 (3')</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>50</td>
<td>No: Corollary 4 (3')</td>
<td>16</td>
<td>20</td>
<td>20</td>
<td>Yes: Theorem 12</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>45</td>
<td>No: Corollary 4 (3')</td>
<td>21</td>
<td>30</td>
<td>35</td>
<td>No: Corollary 4 (4')</td>
</tr>
<tr>
<td>6</td>
<td>40</td>
<td>45</td>
<td>No: Corollary 4 (3')</td>
<td>25</td>
<td>30</td>
<td>30</td>
<td>Yes: Theorem 12</td>
</tr>
<tr>
<td>6</td>
<td>45</td>
<td>50</td>
<td>No: Corollary 4 (4')</td>
<td>25</td>
<td>40</td>
<td>40</td>
<td>Yes: Ash</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>28</td>
<td>Unknown</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 2: Non-GYD PGYD solutions where GYDs do not exist for $v \leq 25, k \leq b \leq 50$

<table>
<thead>
<tr>
<th>$v$</th>
<th>$k$</th>
<th>$b$</th>
<th>GYD?</th>
<th>Non-GYD PGYD?</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>20</td>
<td>50</td>
<td>No: Corollary 5 (2')</td>
<td>Yes: Appendix</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>6</td>
<td>No: Corollary 5 (1')</td>
<td>Yes: Cheng, Theorem 9</td>
</tr>
<tr>
<td>9</td>
<td>30</td>
<td>30</td>
<td>No: Corollary 5 (2')</td>
<td>Yes: Cheng, Theorem 9</td>
</tr>
<tr>
<td>9</td>
<td>42</td>
<td>42</td>
<td>No: Corollary 5 (2')</td>
<td>Yes: Theorem 9</td>
</tr>
<tr>
<td>16</td>
<td>12</td>
<td>36</td>
<td>No: Corollary 5 (1')</td>
<td>Yes: Theorem 9</td>
</tr>
<tr>
<td>18</td>
<td>12</td>
<td>48</td>
<td>No: Corollary 5 (1')</td>
<td>Yes: Appendix</td>
</tr>
<tr>
<td>25</td>
<td>45</td>
<td>45</td>
<td>No: Corollary 5 (2')</td>
<td>Yes: Theorem 9</td>
</tr>
</tbody>
</table>
Appendix

Design for \((v, k, b) = (8, 20, 50)\) is the transpose of the following matrix. Empty space is filled by a \(6 \times 2\) grid of latin squares of order 8.

\[
\begin{array}{cccccccc}
1 & 2 & 7 & 8 & 5 & 1 & 4 & 6 \\
2 & 3 & 8 & 4 & 6 & 2 & 3 & 1 \\
3 & 4 & 5 & 6 & 1 & 2 & 3 & 4 \\
4 & 5 & 2 & 1 & 3 & 4 & 1 & 2 \\
5 & 6 & 1 & 2 & 3 & 4 & 1 & 2 \\
6 & 7 & 8 & 3 & 4 & 5 & 6 & 7 \\
7 & 8 & 2 & 3 & 5 & 6 & 7 & 8 \\
8 & 3 & 5 & 6 & 1 & 2 & 3 & 4 \\
\end{array}
\]
Design for \( (v, k, b) = (9, 24, 48) \) is the transpose of the following matrix. Empty space is filled by a \( 5 \times 2 \) grid of latin squares of order 9.

\[
\begin{array}{cccccccc}
4 & 6 & 5 & 7 & 8 & 9 & 2 & 3 \\
7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 \\
1 & 3 & 2 & 4 & 5 & 6 & 7 & 8 \\
6 & 9 & 8 & 2 & 3 & 5 & 7 & 9 \\
9 & 1 & 4 & 3 & 6 & 7 & 9 & 1 \\
8 & 7 & 1 & 5 & 4 & 2 & 3 & 6 \\
3 & 2 & 6 & 8 & 7 & 4 & 5 & 9 \\
5 & 4 & 3 & 9 & 1 & 8 & 6 & 2 \\
2 & 5 & 7 & 6 & 9 & 1 & 3 & 4 \\
4 & 6 & 5 & 7 & 8 & 9 & 1 & 2 \\
7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 \\
1 & 3 & 2 & 4 & 5 & 6 & 7 & 8 \\
9 & 1 & 4 & 3 & 6 & 7 & 9 & 1 \\
8 & 7 & 1 & 5 & 4 & 2 & 3 & 6 \\
3 & 2 & 6 & 8 & 7 & 4 & 5 & 9 \\
5 & 4 & 3 & 9 & 1 & 8 & 6 & 2 \\
2 & 5 & 7 & 6 & 9 & 1 & 3 & 4 \\
4 & 6 & 5 & 7 & 8 & 9 & 1 & 2 \\
8 & 7 & 9 & 4 & 3 & 5 & 2 & 6 \\
9 & 2 & 7 & 5 & 4 & 3 & 6 & 1 \\
7 & 9 & 2 & 3 & 5 & 4 & 6 & 1 \\
3 & 6 & 5 & 7 & 1 & 8 & 2 & 9 \\
6 & 5 & 3 & 1 & 8 & 7 & 9 & 4 \\
1 & 8 & 6 & 9 & 2 & 4 & 7 & 3 \\
8 & 4 & 1 & 2 & 6 & 9 & 1 & 5 \\
1 & 8 & 4 & 9 & 2 & 6 & 7 & 3 \\
4 & 6 & 5 & 7 & 8 & 9 & 1 & 2 \\
7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 \\
1 & 3 & 2 & 4 & 6 & 5 & 9 & 7 \\
3 & 2 & 6 & 8 & 7 & 4 & 5 & 9 \\
5 & 4 & 3 & 9 & 1 & 8 & 6 & 2 \\
2 & 5 & 7 & 6 & 9 & 1 & 3 & 4 \\
9 & 7 & 4 & 5 & 3 & 2 & 1 & 6 \\
6 & 1 & 8 & 3 & 5 & 7 & 9 & 4 \\
8 & 9 & 1 & 2 & 4 & 6 \\
\end{array}
\]
Design for \((v,k,b) = (18,12,48)\) is the transpose of the following matrix.

\[
\begin{array}{cccccccccccc}
4 & 5 & 6 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
1 & 2 & 3 & 7 & 8 & 9 & 16 & 17 & 18 & 10 & 11 & 12 \\
7 & 8 & 9 & 13 & 14 & 15 & 1 & 2 & 3 & 4 & 5 & 6 \\
13 & 14 & 15 & 16 & 17 & 18 & 4 & 5 & 6 & 7 & 8 & 9 \\
10 & 11 & 12 & 1 & 2 & 3 & 7 & 8 & 9 & 13 & 14 & 15 \\
16 & 17 & 18 & 4 & 5 & 6 & 10 & 11 & 12 & 1 & 2 & 3 \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 16 & 17 & 18 & 4 \\
17 & 18 & 13 & 14 & 15 & 7 & 8 & 9 & 1 & 2 & 3 & 16 \\
14 & 15 & 1 & 2 & 3 & 4 & 5 & 6 & 10 & 11 & 12 & 13 \\
11 & 12 & 4 & 5 & 6 & 13 & 14 & 15 & 7 & 8 & 9 & 10 \\
2 & 3 & 10 & 11 & 12 & 16 & 17 & 18 & 13 & 14 & 15 & 1 \\
8 & 9 & 16 & 17 & 18 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
3 & 4 & 2 & 15 & 7 & 5 & 18 & 10 & 8 & 12 & 13 & 17 \\
6 & 10 & 5 & 3 & 1 & 8 & 9 & 13 & 11 & 18 & 16 & 14 \\
15 & 1 & 14 & 6 & 4 & 2 & 12 & 16 & 17 & 9 & 7 & 11 \\
9 & 13 & 17 & 18 & 10 & 11 & 3 & 7 & 14 & 6 & 4 & 2 \\
12 & 16 & 8 & 9 & 13 & 17 & 6 & 1 & 2 & 15 & 10 & 5 \\
18 & 7 & 11 & 12 & 16 & 14 & 15 & 4 & 5 & 3 & 1 & 8 \\
2 & 3 & 5 & 6 & 7 & 8 & 10 & 11 & 13 & 15 & 16 & 18 \\
1 & 17 & 3 & 8 & 6 & 9 & 11 & 12 & 4 & 13 & 14 & 16 \\
4 & 1 & 2 & 5 & 15 & 7 & 12 & 10 & 14 & 18 & 9 & 17 \\
3 & 2 & 9 & 7 & 5 & 6 & 14 & 16 & 10 & 12 & 17 & 13 \\
8 & 5 & 1 & 2 & 4 & 11 & 9 & 13 & 12 & 16 & 18 & 15 \\
7 & 8 & 4 & 1 & 3 & 17 & 6 & 18 & 15 & 14 & 10 & 11 \\
18 & 16 & 14 & 13 & 12 & 10 & 8 & 9 & 5 & 4 & 3 & 2 \\
14 & 9 & 17 & 11 & 10 & 16 & 15 & 7 & 6 & 5 & 1 & 3 \\
15 & 13 & 11 & 17 & 2 & 12 & 18 & 8 & 7 & 1 & 6 & 4 \\
9 & 11 & 10 & 16 & 17 & 15 & 13 & 4 & 3 & 2 & 8 & 6 \\
10 & 18 & 12 & 15 & 16 & 14 & 7 & 5 & 8 & 6 & 2 & 3 \\
11 & 12 & 7 & 9 & 14 & 18 & 4 & 3 & 1 & 17 & 13 & 5 \\
12 & 7 & 13 & 4 & 11 & 5 & 16 & 17 & 2 & 3 & 15 & 9 \\
17 & 10 & 6 & 14 & 13 & 3 & 5 & 1 & 18 & 8 & 12 & 7 \\
6 & 15 & 16 & 10 & 18 & 1 & 2 & 15 & 11 & 9 & 4 & 8 \\
13 & 4 & 18 & 12 & 8 & 2 & 3 & 6 & 16 & 7 & 11 & 14 \\
5 & 6 & 15 & 18 & 9 & 13 & 1 & 2 & 17 & 11 & 7 & 10 \\
16 & 15 & 8 & 3 & 1 & 4 & 17 & 14 & 9 & 10 & 5 & 12 \\
2 & 3 & 4 & 5 & 7 & 9 & 10 & 11 & 15 & 16 & 18 & 14 \\
1 & 6 & 5 & 3 & 11 & 7 & 12 & 8 & 13 & 15 & 16 & 17 \\
4 & 1 & 2 & 6 & 12 & 10 & 8 & 9 & 14 & 13 & 17 & 18 \\
3 & 2 & 6 & 4 & 10 & 8 & 7 & 12 & 17 & 14 & 15 & 16 \\
6 & 5 & 1 & 2 & 9 & 12 & 11 & 7 & 16 & 18 & 14 & 13 \\
5 & 4 & 3 & 1 & 8 & 11 & 9 & 10 & 18 & 17 & 13 & 15 \\
8 & 12 & 11 & 9 & 14 & 17 & 15 & 18 & 2 & 5 & 3 & 6 \\
12 & 8 & 9 & 11 & 17 & 14 & 18 & 15 & 5 & 2 & 6 & 3 \\
7 & 9 & 10 & 12 & 15 & 18 & 13 & 16 & 3 & 6 & 1 & 4 \\
9 & 7 & 12 & 10 & 18 & 15 & 16 & 13 & 6 & 3 & 4 & 1 \\
11 & 10 & 7 & 8 & 13 & 16 & 14 & 17 & 1 & 4 & 2 & 5 \\
10 & 11 & 8 & 7 & 16 & 13 & 17 & 14 & 4 & 1 & 5 & 2 \\
\end{array}
\]

References


