Characterization of discretely graded materials using acoustic wave propagation

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Abstract

Functionally graded materials (FGMs) have great potential as energy absorbing devices, electrical transducers, thermal barriers, etc. Study of the propagation of elastic waves is imperative for a number of such applications. In this paper, a simple one-dimensional model is proposed to study the stress waves in discretely graded media. The model uses spectral approach to determine the stresses due to the incident and reflected waves in FGMs. It has been observed that FGMs attenuate and delay the peak stresses considerably as compared to composites with sharp interfaces. Performance of different number of inner layers on the reflected wave characteristics has been discussed. Stress time history profiles are also presented for Alumina–Aluminum FGMs that are used in armor applications.

Keywords: Functionally graded materials; Wave mechanics; Spectral method; Peak stresses; Time delay

1. Introduction

In the last decade, there has been considerable interest in grading the properties of composites by altering the volume fractions of their constituents as a tool for designing new materials with contradicting performance requirements [1]. Most of the research in this area involves the development of graded coatings and interfacial regions for the purpose of reducing residual and thermal stresses and increasing the bonding strength. Such composites with continuously varying volume fractions are known as functionally graded materials (FGM). Thermo-electro-mechanical characterization of these materials is imperative for proper exploitation of the superior properties offered by them. Propagation of elastic waves is one of the very promising ways for such characterization. Moreover, FGMs are frequently employed for the assimilation of mechanical [2], thermal [3], and electrical [4] shocks and disturbances. The performance evaluation is possible through the study of the propagation of waves through such media. In elastodynamics of materials with continuously varying properties, the pulse shape is distorted in time, the wave propagation speed is not constant, and there are no sharp interfaces that would cause wave reflections. Consequently, even in the simple case of one-dimensional wave propagation the locations and magnitudes of peak stresses cannot be determined by inspection.

Bruck [2] has proposed a simple one-dimensional model for the analysis of stress waves in discretely layered FGMs. The model was also empirically extended to continuous FGMs, Vasudeva and Bhaskara [5] explained the propagation of pressure pulse in a non-homogenous elastic rod with varying cross-section by a method of Keller and Keller [6]. Gopalakrishanan and chakraborty [7] have analyzed wave propagation behaviour in FGM beams for impulsive loading.

In this paper, a simple one-dimensional model is formulated for analyzing the stresses due to reflected waves in discretely layered FGMs. It is assumed that the waves propagate in one direction only. The reflected stress peaks
are determined using the spectral approach [8]. The spectral method is discussed later in the paper.

2. Modelling of discrete FGM

FGMs can be modelled as shown in Fig. 1. It consists of discrete layers of gradually varying material property. The left side of the discrete FGM consist of base material 1 and on the right side of it is base material 2. In between these two materials, discrete layers of intermediate material properties are present. The discretely layered FGM can be infinitely refined to form a continuous FGM. The physical properties of these inner layers are assumed to follow linear rule of mixtures. It is assumed that stress waves are linear, elastic, longitudinal waves propagating in one direction only.

Recently Bruck [2] has formulated a one-dimensional model by tracing the path of the reflected and transmitted waves for discretely graded FGMs. The stresses in reflected wave coming back to base material 1 (Fig. 2) are given as

\[
\sigma_N = \frac{(1 - z_0)}{(1 + z_0)} + \sum_{j=1}^{m} \left[ \frac{(1 - z_j)}{(1 + z_j)} \prod_{k=1}^{j} \frac{4z_{k-1}}{(1 - z_{k-1})^2} \right] + \text{HOTs}
\]

where, \(\sigma_N\) is the normalized stress and \(z_j\) is the ratio of acoustic impedance of layer \(j + 1\) to \(j\), and \(\text{HOTs}\) are higher order terms comprised of three or more reflections and one or more transmissions.

In addition, the normalized time for the wave reflected from the \(j\)th layer to reach base material 1 is given by

\[
\bar{t} = \frac{t}{(d/c_0)} = \frac{2}{m} \sum_{k=1}^{m} c_0
\]

where \(\bar{t}\) is the time waves take to reach base material 1 (Fig. 2), \(c_0\) is the wave speed in base material 1, \(m\) is number of discrete layers, \(d\) is discrete FGM thickness and \(c_k\) is the wave speed in layer \(k\). The \(\text{HOTs}\) are of the form

\[
(-1)^p \left[ \frac{(1 - z_j)}{(1 + z_j)} \right]^{(2p+1)} \left\{ \frac{4z_j}{(1 + z_j)^2} \right\}^{(2q+1)} = (-1)^p R T
\]

where \(p \geq 1\) and \(q \geq 1\)

where \(-1 < R < 1\) represents the product resulting from each reflection that the propagating stress waves experiences at a sharp interface between layers in the graded interface. Although this method is simple and effective as a design tool, it cannot be extended to higher dimensions. A generalized method such as finite element method is capable of modeling such cases. However, requirement of very fine mesh and small time steps pose a serious challenge to the viability of the method. In this paper, we adopt the spectral method for studying wave propagation in FGMs.

3. Spectral analysis

The characteristics of wave propagation problems are that the frequency content of the exciting force is very high. Therefore, a very fine mesh of finite elements is necessary to adequately model the problem. This problem can be alleviated if we use frequency based methods, such as the spectral method [8] instead of the time based techniques. In this method, the governing partial differential wave equation is reduced to a set of ordinary differential equations. Their solution is easier than the original differential equation. However, often approximate solutions are sought. The transformation is effected by Fast Fourier Transform (FFT) Algorithm. These solutions to the governing equations are used as shape functions for spectral element formulation. The advantage of this method is that the inertial effects are exactly represented and hence often exact solutions are obtained. In addition, often the resulting element is super convergent and very few elements are required to model the system.
3.1. Reconstruction of waves

The significance of the spectral approach to waves along with differential equation is that once the signal is characterized at one position in space then it is known at all positions, and therefore, propagating it becomes a fairly simple matter. The solution obtained by the spectral method is in frequency domain and to get back to the time domain inverse FFT is done. It should be observed that while using FFT for inversion the transformed solution in frequency domain is calculated up to Nyquist frequency and the rest is obtained by imposing the condition that it must be the complex conjugate of the initial part. This is done to ensure that the reconstructed history is real.

3.2. Spectral approach

The governing wave motion equation is given by

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

where, $E$ is the elastic modulus, $\rho$ is the mass density and $A$ is the cross-sectional area of the rod. Considering both the modulus and the area do not vary with position, the displacement solution to the above equation for rod of length $L$, according to spectral approach [8], can be represented by

$$u(x, t) = \sum_n \hat{u}_n(x, \omega_n) e^{i\omega_n t}$$

where $\omega_n$ represents the frequency and the spectral displacements $\hat{u}_n$ has the following solution:

$$\hat{u}_n = Ae^{-ikx} + Be^{ik(L-x)}$$

where $k$ is the wave number and $A$ and $B$ are undetermined amplitudes at each frequency, determined by applying boundary conditions. Hence, the displacements in the two base materials for sharp interface (with no inner layer present between the two base materials) can be written as

$$\hat{u}_1 = A_1 e^{-ikx} + B_1 e^{ikx}$$

$$\hat{u}_2 = A_2 e^{-ikx}$$

Here, $\hat{u}_1$ and $\hat{u}_2$ are displacements in respective mediums, $k$ is the corresponding wave number, and $A_1$, $B_1$ and $A_2$ are parameters determined by applying appropriate boundary conditions. The first base material will have both incident and reflected waves while second base material will only have transmitted waves. The stress in reflected wave and transmitted waves can be obtained by applying force equilibrium and displacement compatibility conditions at the point of interface. Since the area ($A$) of both the base materials is assumed to be same, transmitted and reflected stresses in terms of incident stress can be determined as follows:

$$\sigma_i = \frac{\sqrt{E_1 \rho_2 / E_1 \rho_1} - 1}{\sqrt{E_1 \rho_2 / E_1 \rho_1} + 1} \sigma_i$$

$$\sigma_t = \frac{2 \sqrt{E_1 \rho_2 / E_1 \rho_1}}{\sqrt{E_1 \rho_2 / E_1 \rho_1} + 1} \sigma_i$$

where $\sigma_i$ is the amount of stress in incident wave, $\sigma_t$ is the amount of stress in transmitted wave and $\sigma_r$ is the amount of stress in reflected wave. $E$ is the modulus of elasticity and $\rho$ is the density of the corresponding media. The ratio of impedances of the base materials determines the reflected and the transmitted stress.

Similar expressions can be obtained for discretely graded FGMs. For FGMs with one inner layer between the two base materials, the displacements in the base materials and the inner layer can be written as

$$\hat{u}_1 = A_1 e^{-ikx} + B_1 e^{ikx}$$

$$\hat{u}_2 = A_2 e^{-ikx} + B_2 e^{ikx}$$

$$\hat{u}_3 = A_3 e^{-ikx}$$

Applying the boundary conditions at the interfaces ($x = 0$ and $x = d$), the motions can be expressed in matrix form as

$$\begin{pmatrix} p_1 & p_2 & 0 \\ 0 & e^{-ikx} & -e^{-ikx} \\ 0 & -p_2 e^{-ikx} & p_2 e^{-ikx} \\ \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

where, $p_n = (kE_1)_n$

Hence, the transformed solution in frequency domain can now be obtained in terms of the incident wave. To get back in time domain inverse FFT is required. If the FGM consists of $m$ inner layers between the two base materials then all the parameters can be obtained in a similar manner. The assembled matrix then will be a square matrix of size $2m + 2$. Hence, it is possible to determine the reflected and transmitted stresses in each of the inner layers. Several cases that explain the utility of the proposed model are discussed in the next section.

4. Numerical results

In this section, we carry out the validation test of the proposed method and the software and then apply the method to a number of problems of interest.

4.1. Validation problem

Bruck [2] developed expressions (Eqs. (1)–(3)) for reflected peak stress and delay times of discretely layered material system defined by their ratio of acoustic impedances. For an acoustic impedance of material 2 to material 1 is 0.01 the results of Bruck have been compared with the present results (Table 1). The graded layer thickness is 1 mm. The input impulse is applied at 0.001 s.
reflected stress peaks and the time delay observed for one inner layer with linear gradation are presented. It may be noted that the HOTs (Eq. (3)) have been included in the calculations. The various peak stresses and delay time are obtained from the two methods are identical.

4.2. New problems

The proposed model is applied to a number of problems with variable material grading. For all the cases the same triangular sharp impulse (Fig. 3a) is applied on base material 1 (Fig. 2). Since the spectral approach is a frequency based method, impulse FFT (Fig. 3b) acts as input. The thickness \((d)\) of the graded layer is 2 mm in all cases. The reflected stress is measured at the interface \((x = 0)\) of base material 1 and the first graded layer (Fig. 4). The composite discussed here is Alumina–Aluminum FGM. This combination is of special interest due to its energy absorbing capability. The physical properties of the FGM are in Table 2. The stress time history of reflected waves has been

<table>
<thead>
<tr>
<th>Time ((\mu s))</th>
<th>Peak stresses ((x \times 10^{10} \text{ N/m}^2))</th>
<th>Time delay (s)</th>
</tr>
</thead>
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<tr>
<td></td>
<td>Bruck Present Bruck Present</td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>−0.328903 −0.328908</td>
<td>0.00E+00 0.00E+00</td>
</tr>
<tr>
<td>1.3</td>
<td>−0.857188 −0.857193</td>
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<tr>
<td>1.5</td>
<td>0.270983 0.271000</td>
<td>8.00E−07 8.00E−07</td>
</tr>
<tr>
<td>1.7</td>
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<td>1.20E−06 1.20E−06</td>
</tr>
<tr>
<td>1.9</td>
<td>−0.027081 0.027195</td>
<td>1.60E−06 1.60E−06</td>
</tr>
<tr>
<td>2.1</td>
<td>−0.008561 −0.008889</td>
<td>2.00E−06 2.00E−06</td>
</tr>
</tbody>
</table>

Table 1
Comparison of present and Bruck’s [2] results

Fig. 3a. The sharp impulse applied to base material 1.

Fig. 3b. FFT of the incident impulse.
plotted. The absolute stresses have been presented to enable comparison of magnitude only.

Discrete layered FGM consist of many layers, whose physical properties are intermediate of that of the base materials properties. These inner layers graded according to the power law and complete polynomial are discussed here. For power law variation the volume fraction of the base materials \( 1 \) and \( 2 \) at \( x \) (Fig. 4) can be written as

\[
V_2^i = \frac{x_i^n}{D} C_1 C_7^n \quad (14)
\]

\[
V_1^i = 1 - V_2^i \quad (15)
\]

where \( V_i \) is the volume fraction of the respective base materials in the \( i \)th layer, \( x_i \) is the coordinate of the center of the layer and \( n \) is an exponent that can varied as required. The variation of \( E \) and \( \rho \) for Alumina–Aluminum FGM within the graded layers for different power law exponent \( n \) is shown in Fig. 5.

In case of power law the variation in the material properties is controlled by a single parameter \( n \). One gets the opportunity of additional controls if instead of power law the complete polynomial expressions are used to define the gradings

\[
V_2^i = \sum_{j=0}^{n-1} a_j \left( \frac{x_i}{D} \right)^j \quad (16)
\]

where \( n \) is the order of the curve and \( a_j \) is the unknown coefficient determined from the boundary conditions. Thus, in addition to the boundary values the slopes of the gradings can be controlled through these equations. Fig. 6 shows the variation of \( E \) and \( \rho \) in the graded material for complete quadratic and cubic polynomial variation.

### 4.2.1. Case I: Sharp interface

In sharp interface, there is no inner layer between the two base materials. Eq. (9) can be used for determining stresses in the reflected and the transmitted waves. Fig. 7a shows the reflected stress time history for the sharp interface. A single peak is observed due to the reflection from the interface of the two base materials.

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**Table 2**

<table>
<thead>
<tr>
<th>Property</th>
<th>Alumina</th>
<th>Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus (GPa)</td>
<td>400</td>
<td>70</td>
</tr>
<tr>
<td>Density (kg/m(^3))</td>
<td>4000</td>
<td>2700</td>
</tr>
<tr>
<td>Acoustic impedance (kg/m(^2)/s)</td>
<td>1265</td>
<td>435</td>
</tr>
</tbody>
</table>

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**Fig. 4.** Configuration of discretely graded FGMs showing power law variation.

**Fig. 5.** Density and elastic modulus variation in the graded layer for polynomial gradations.

**Fig. 6.** Density and elastic modulus variation in the graded layer for polynomial gradations.
magnitude of the reflected stress is about half of that of the incident stress. Time taken for the wave to get reflected back to base material 1 is zero, since we are measuring reflected stress at the interface only \((x = 0)\). Although only about 49% of the incident stress is reflected back. This agrees with the acoustic impedance ratios of the material combination. Rest of the energy is transmitted to base material 2.

The elegance of the proposed model is that the movement of the waves in the material can be examined with time. The stress wave that is transmitted in the base material 2 can be plotted at different coordinates (Fig. 7b). The same time gap between the stress peaks can be explained by the uniform velocity of the wave propagating in Aluminium.

4.2.2. Case II: Power law gradation

The sharp interface is eliminated by graded layer(s) between the two base materials. The variation of material properties in the inner layer(s) is assumed to follow the power law. Fig. 8a shows the reflected stress time history for one inner layer for linear variation. This is obtained using Eq. (13). Only three significant peaks are observed since after four reflections the magnitude of the reflected

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**Fig. 7a. Stress time history for sharp interface.**

**Fig. 7b. Stress time history for transmitted wave at different locations in base material 2.**

**Fig. 8a. Reflected stress time history for one inner layer (linear variation).**
stress is reduced to almost 1% of the incident stress. The peak reflected stress is also reduced to about 25% of the incident stress wave. The time taken to attain peak stress is delayed too. The time delay defers the onset of damage. The peaks are labelled to show the number of reflections coming back to base materials. Figs. 8b and 8c show the stress time history for 2 and 4 inner layers respectively. It is noticed that as the number of inner layers increases the peak reflected stress goes down. This is expected since the number of reflections increases with the increase in the number of inner layers.

Other power law variations are examined for their energy absorbing potential. Fig. 9a depicts the peak reflected stresses for different power law variations with the varying number of inner layers. It can be observed that with the increase in the number of inner layers the peak stresses come down for all the cases. The acoustic impedance ratios for the material systems can be studied to understand the phenomenon. Fig. 9b shows the acoustic impedance ratios for varying number of inner layers for $n = 2$. The impedance ratios increases, reaches near to 1 and the slopes of the curves reduce with the increase in the number of inner layers. As the ratio comes near to 1, the reflected stress reduces as clear form Eq. (9). Ratio near to 1 shows gradual changes in the material properties. The reflected stress increases in case the ratio vary away from 1. It may be either greater or lesser than 1. It can be explained from the material properties point of view. Ratio near to 1 depicts that the material properties changes very slowly and hence all the waves will transmit, since waves will expe-
rience almost the same material when travelling from one
layer to another. However, manufacturing constraints
may limit the number of inner layers. Therefore, a judicious
design decision can be arrived at by choosing the number
of inner layers depending on the target reduction of peak
stress.

When we compare the peak stresses for different expo-
nents \( n \) for the same number of inner layers it is clear that
\( n = 0.5 \) gives the least peak stress. The reflected peak stress
was only about 11% of the applied peak stress in case of
four inner layers. We shall once again study the acoustic
impedance ratios to analyze these results. Fig. 9c shows
the variation of impedance ratios with varying \( n \) for four
inner layers FGM. We observe that in case of \( n = 0.5 \) the
slopes of the curve are the lowest. In other words, the
impedances in the case of \( n = 0.5 \) have varied the least.
The increased variations in the impact ratios also increase
the variation in stress peaks. It may be noted that \( n = 0.5 \)
results in a quadratic curve and its second derivative is con-
stant (in the present case in a discretized sense). Therefore,
the grading that has uniform, or most uniform, rate of
change of properties will result in the lowest peak stresses.

The time delay histograms are plotted for these grada-
tions in Fig. 10. Intuition suggests that the grading that
has the maximum slope away from the point of measure-
ment will have the highest delay times. That means the
delay time should increase with the exponent \( n \). However,
the results do not follow this conjecture. For higher values
of \( n \) the reflected waves travel through a stiffer medium
resulting in higher velocities. As a result, the delay times
do not follow the expected path and for exponents higher
than 2 the delay times reduce marginally. Exponent of 2
has given the highest delay time. However, the delay time
for \( n = 0.5 \) was not much lower. Therefore, considering
the peak stress benefit, \( n = 0.5 \) seems to be the ideal grading
for this system.

![Fig. 9b. Impedance ratios for n = 2.](image)

![Fig. 9c. Impedance ratio for four inner layers.](image)

![Fig. 11a. Peak reflected stress for polynomial variations.](image)

![Fig. 11b. Time delay for polynomial variation.](image)
4.2.3. Case III: Polynomial gradation

Polynomial variations are also analyzed for their energy absorbing ability. Figs. 11a and 11b illustrate peak reflected stresses and time delay benefits for the complete quadratic and cubic polynomial variation with the increasing number of inner layers. The quadratic grading reduces the peak stresses more than the cubic grading due to its more uniform inter-layer acoustic impedances. The delay times are higher in the cubic grading than the quadratic grading. The power law grading offers better results as compared to polynomial gradation.

Finally, it should be kept in mind that the above observations are based on Alumina–Aluminum FGM. Although the performance of an FGM would depend on the physical properties of the base materials the observations should hold good for any material combination where the impulse travels from the stiffer to the softer material and the point of interest is at the interface of the stiffer material and the FGM.

5. Concluding remarks

A simple one-dimensional model based on the spectral approach has been proposed for material characterization and managing stress waves in discrete FGMs. The model is accurate and does not lack generality. Following conclusions can be drawn from the present investigation:

1. Stress wave magnitudes attenuate very fast in the graded media as compared to the sharp interface structure.
2. Graded media also delays the peak stress arrival, which should delay the time of damage initiation.
3. The stress time histories are highly sensitive to the type of grading and base material properties.
4. An FGM with power law grading exponent of 0.5 is best suited for the reduction of peak reflected stress.
5. The best time delay is achieved for the exponent of 2.0.
6. The complete polynomial gradings of the second and the third order were not superior to the power law gradings in the present case.
7. Overall the performance of $n = 0.5$ was found to be most satisfactory.

References