Reciprocity relations for particle mixtures

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Several phenomena in particle physics involve the transformation of one kind of particle into another. This paper investigates the general conditions under which the rate of a particular transition can be proportional to that for its inverse, as was found by explicit calculation for the case of neutral $K$ mesons. For the case of neutrinos, constancy of the ratio can occur only if that ratio is unity; this may offer a more convenient test of time-reversal invariance than a direct comparison of reciprocal transition rates.

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I. INTRODUCTION

Comparison of the rate for $K^0 \rightarrow \bar{K}^0$ transitions with that for the inverse process $\bar{K}^0 \rightarrow K^0$ is currently under study [1] to check the predicted [2] departure from $T$ invariance. The ratio $R$ of the two rates was calculated according to the current theory and found [2] to be constant in time even though the individual transition rates are expected to show significant time dependence. This is now understood as a general feature of all theories, such as the generalized Weisskopf-Wigner theory, in which the decay of two mixing states can be described by particular superpositions which decay independently.

In Sec. II, we extend the discussion to a larger number of mixing states and find that the constancy of cross ratios can be maintained even when the number of mixing states exceeds 2. For the particular case of $n = 3$, one finds the interesting result that, if any one cross ratio is time independent, the other two will automatically be forced to be time independent as well. In Sec. III, this analysis is applied to the currently interesting problem of neutrino oscillations and shown to simplify the tests of time-reversal invariance proposed for such phenomena. Section IV summarizes our conclusions.

II. GENERAL PARTICLE-MIXTURE THEORY

Although it was not explicitly mentioned in Ref. [2], the constancy of

$$ R = \frac{P_{K^0 \rightarrow \bar{K}^0}(t)}{P_{\bar{K}^0 \rightarrow K^0}(t)} $$

found there did not depend on the assumption of exponential time dependence of $K_S$ and $K_L$ states (adopting the current notation for the short-lived and long-lived neutral kaon states). All that is required is the assumption that the time evolution of any neutral kaon state (consisting of an arbitrary superposition of $K^0$ and $\bar{K}^0$ states) is determined by reexpressing that state as a linear combination of two particular superpositions $K_S$ and $K_L$, which have the property that the relative amplitude of $K^0$ and $\bar{K}^0$ components does not change with time, although the overall amplitude for each varies in a determined (and distinct) way [3]. Subject to this proviso [4], the assumption of a two-level structure is sufficient to guarantee that $R$ is independent of time. Any observed deviation from constancy would then indicate the presence of another state mixing with $K^0$ and $\bar{K}^0$.

While the assumption of (only) two mixing states suffices to assure the constancy of $R$ (which must be unity if time-reversal invariance is valid) one may ask whether it is also a necessary condition. We shall see that this is not the case, i.e., the ratio $R$ can be independent of time (even if it is not equal to unity) also when the number of mixing states exceeds 2.

Suppose that there are $n$ "flavor" states $F_j$ which mix with each other in such a way that $n$ particular linear superpositions $S_\mu$ of these states evolve independently of each other, varying only in overall amplitude (and phase) in the course of time:

$$ S_\mu = \sum_{j=1}^{n} D_{\mu j} F_j. $$

If, as we shall assume, $D$ is nonsingular, we may write

$$ F = D^{-1} S $$

in a compact notation. Let us assume that $S_\mu$ evolves according to

$$ S_\mu \overset{t}{\rightarrow} \theta_\mu(t) S_\mu, $$

where $\theta_\mu(t)$ is a given (unspecified) function of time. According to Eqs. (3) and (4), a state created initially as $F_j$ will be found at a later time to have transformed into

$$ F_j = D^\dagger \theta_\mu(t) D S_\mu, $$

where $D^\dagger$ is the Hermitian transpose of the matrix $D$. For the particular case of Sec. I, $S_\mu$ and $\theta_\mu(t)$ are simply

$$ S_1 = \sin \alpha F_1 + \cos \alpha F_2, $$

$$ S_2 = \cos \alpha F_1 - \sin \alpha F_2, $$

$$ \theta_1(t) = \exp \left[ -\frac{g}{\hbar} t \right], $$

$$ \theta_2(t) = \exp \left[ \frac{g}{\hbar} t \right], $$

where $g$ is the weak neutral current coupling constant. For the case of neutrinos

$$ g = \frac{G_F}{\sqrt{2}} |\bar{\nu}_e| |\nu_\mu|, $$

where $G_F$ is the Fermi constant and $|\bar{\nu}_e|$ and $|\nu_\mu|$ are the transition amplitudes.
Consequently, if \( r_{ij}(\lambda) \) and \( r_{ik}(\lambda) \) are found to be independent of \( \lambda \), it necessarily follows that \( r_{jk}(\lambda) \) is independent of \( \lambda \). Therefore, for any value of \( n \), to make all cross ratios \( R_{jk} \) time independent, it suffices to require all \( R_{ij} \) \((j = 2 \text{ to } n)\) to be independent of time. But we have seen that Eq. (10) forces \( r_{jk}(n) \) to have the same value \( \rho_{jk} \) as the remaining \( r_{jk}(\lambda) \), \( \lambda = 1 \text{ to } (n - 1) \), if these latter are required to assume the common value \( \rho_{jk} \). Therefore constancy of all \( R_{jk} \) is assured by \( \lambda \) independence of all \( r_{jk}(\lambda) \) for each value of \( k \) from 2 to \( n \), and for \( \lambda \) from 1 to \( (n - 1) \). Writing these conditions in the form

\[
\rho_k = r_{ik}(\lambda) \quad \text{for } \lambda = 1 \text{ to } (n - 1)
\]

for every value of \( k \) from 2 to \( n \), we see that there are altogether \((n - 1) \times (n - 2)\) such conditions, corresponding to the \((n - 2)\) conditions for each of the \((n - 1)\) possible values of \( k \). We now show that \((n - 2)\) of these will be identically satisfied. For this, let us rewrite Eq. (12), for a given value of \( k \), using the explicit form of \( r_{jk} \) from Eq. (8):

\[
\rho_k = \frac{D_{\gamma k}^{-1}D_{\mu k}}{D_{\gamma i}^{-1}D_{\mu i}^{-1}} = \frac{D_{\gamma k}^{-1}D_{\nu k}}{D_{\gamma i}^{-1}D_{\nu i}^{-1}}
\]

for \( \mu \neq \nu \) running from 1 to \((n - 1)\). Equation (13) can be rewritten as

\[
\frac{D_{\gamma k}D_{\mu k}}{D_{\nu k}} = \frac{D_{\gamma i}D_{\nu i}}{D_{\mu i}}.
\]

There will be one such equation for each value of \( k \); thus if we define the left-hand side (LHS) as \( t_{\mu \nu}(k) \), Eq. (13') expresses the condition that \( t_{\mu \nu}(k) \) should not depend on \( k \). But, from its definition, \( t_{\alpha \beta}(k) \) satisfies

\[
t_{\alpha \beta}(k) = t_{12}(k)/t_{13}(k);
\]

therefore it suffices to require \( k \) independence of all \( t_{\mu \nu}(k) \) to assure \( k \) independence of all \( t_{\mu \nu}(k) \). By summing the numerators and denominators, respectively, of the LHS of Eq. (13') for all \( k \) from 2 to \( n \), we obtain quantities which, by virtue of \( D \cdot D^{-1} = I \), are identically equal to the numbers appearing on the RHS. This effectively reduces the number of \( k \) values to be considered by 1, leaving us with \((n - 2)^2\) independent conditions to be satisfied.

For the case \( n = 3 \), therefore, it follows that a single equation suffices to guarantee that all three cross ratios \( R_{12}, R_{23}, \) and \( R_{31} \) will be constant. As a corollary, it follows that constancy of any one cross ratio will force the other two to be constant as well. An arbitrary \( 3 \times 3 \) mixing matrix can be written in the form

\[
D_3 = \begin{bmatrix}
\alpha_1 & \alpha_1 \beta_1 & \alpha_1 \gamma_1 \\
\alpha_2 & \alpha_2 \beta_2 & \alpha_2 \gamma_2 \\
\alpha_3 & \alpha_3 \beta_3 & \alpha_3 \gamma_3 \\
\end{bmatrix},
\]

for which one then finds

\[
\begin{bmatrix}
1 & \beta_1/\gamma_1 \\
1 & \beta_2/\gamma_2 \\
1 & \beta_3/\gamma_3 \\
\end{bmatrix} = 0
\]

(15)

to be the condition for constancy of all cross ratios.
$R_{12}, R_{23},$ and $R_{31}$. Equation (15) defines the family of mixing parameters for which this property holds.

III. APPLICATION TO NEUTRINOS

Our analysis has shown that the constancy of the cross ratios $R_{jk}$ is not assured in general for $n \geq 3$. High-energy physics experiments have shown that three different neutrino species are required [8] to account for the observations. The application to possible neutrino mixing [9], therefore, becomes interesting, especially in the context of the suggestion that $T$ noninvariance may manifest itself in this phenomenon. We emphasize that the following discussion is valid for arbitrary $n$.

In the usual analysis [10] of neutrino mixing, the matrix $D$ is taken to be unitary. For unitary $D$, the denominator of Eq. (8) becomes the complex conjugate of the numerator, and thus $r_{jk}$ necessarily has modulus unity. If we equate the RHS for any two values of $\lambda$, we see that the phase of the numerator (or of the denominator) must be the same for both cases. It follows that, if $D$ is unitary, the constancy of $R_{jk}$ is possible only if its value is unity, which is the requirement of reciprocity. Whereas a direct test of reciprocity requires a knowledge of relative normalizations for the to and fro reactions, it is obviously easier to test for the time independence of the ratio, which does not require this information. A corollary statement is that if $R_{jk}$ were found to be constant but not equal to 1, then the unitarity of $D$ would be in question.

For the case of $n = 3$, the analysis of Sec. II for general $D$ showed that constancy of any one cross ratio $R_{jk}$ assured the constancy of the other two. For unitary $D$, it now follows that the validity of the reciprocity relation for any pair of neutrino species will require that reciprocity hold also for the other two pairs [11].

Thus far, it was not necessary to assume any particular form $\theta_k(t)$ for the time-evolution factors. For the particular case of stable neutrinos, these necessarily have the form $\theta_k = \text{exp}(\mp im_k t)$, with $m_k$ real. In that case $\theta_k^* = \theta_k(-t)$ and, for unitary $D$, Eq. (6) yields

$$A_{jk}^*(t) = A_{kj}(-t),$$

leading directly to the relation

$$\omega_{jk}(t) = \omega_{kj}(-t)$$

between the rate of transitions from $\nu_k$ to $\nu_j$ and vice versa. Combining this with our finding $R_{jk} = 1$, viz.,

$$\omega_{jk}(t) = \omega_{kj}(t),$$

for this case, one finds

$$\omega_{jk}(t) = \omega_{jk}(-t),$$

which is the relation given by Cabibbo [12] on the basis of time-reversal invariance. In deriving Eqs. (18) and (19) from the constancy of $R_{jk}$, we only required the phase of $r_{jk}(\lambda)$, defined in Eq. (8), to be independent of $\lambda$, whereas the derivation [12] from time-reversal invariance imposed the apparently stronger condition of reality for all $D_{jk}$. In fact, the two requirements are equivalent because a $\lambda$-independent phase of $r_{jk}$ can always be removed by redefinition of the phases of $F_j$ and $F_k$. We conclude that, for mixing described by a unitary transformation, the reciprocity relations for neutrino transformations, which are required by time-reversal invariance, also follow from the independent, and possibly simpler, requirement that the corresponding cross ratios $R_{jk}$ be time independent. From an experimental point of view, it may be easier to test the time independence of a cross ratio than to perform the tests of Eq. (19) suggested in Ref. [12].

IV. CONCLUSIONS

This paper has analyzed and found the general conditions under which the time dependence of transitions from a particle of type $j$ into one of type $k$ can be proportional to that for the inverse transitions $k \rightarrow j$, under the assumption that certain linear superpositions of these states evolve independently. If $T$ invariance is assumed, the two rates would, of course, have to be equal; we do not make that restrictive assumption. The analysis explains why, for the case when there are only two such states which mix, the (time-dependent) rate for $a \rightarrow b$, which need not equal that for $b \rightarrow a$ if $T$ invariance fails, is nevertheless found [2], by explicit calculation in the generalized Weisskopf-Wigner approximation, to be a constant multiple of it. For three or more mixing states, the constancy of the cross ratios is not expected in general, but can occur in a nontrivial way if certain conditions are satisfied. In the case when the number of mixing states is $n = 3$, it is found that a single condition (on the mixing matrix which determines the composition of the independently propagating states) suffices to assure the constancy of all three cross ratios; this condition is explicitly found. For a larger number of mixing states, we have found that more conditions must be satisfied, which we have expressed in a general form. While the method of analysis presented in this paper applies to arbitrary $n$, explicit discussion of cases with $n > 3$ does not seem to be called for at the present time.

For the application to neutrino mixing we assumed, as usual, that the mixing matrix $D$ is unitary. Then, regardless of the number of types of neutrinos which mix, the constancy of any cross ratio requires that it have the value 1 required by reciprocity. Thus, while the original proposal [12] to test $T$ invariance had called for a test of reciprocity in the transitions from one neutrino type to another, our analysis shows that, for the case of unitary mixing (as assumed also in Ref. [12]), exactly the same information is obtained by checking the constancy of any cross ratio, which obviates the need for knowing the relative normalization for the two transition rates. The experimental advantage of the latter proposal should be apparent.

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[1] CPLEAR experiment, as reported, for example, by A. Schopper, in Results and Perspectives in Particle Physics, Proceedings of the 7th Rencontre de Physique de la Vallee d'Aoste, La Thuile, Italy, 1993 (unpublished).


[3] In the usual Weisskopf-Signer approximation, adapted to the neutral kaon problem by T.D. Lee, R. Oehme, and C.N. Yang, Phys. Rev. 106, 340 (1957), this is accommodated by replacing real energy eigenvalues by complex ones. As we have stated, it is not necessary to resort to this approximation to deduce the constancy of $R$, which would be unaffected by nonexponential decay of $K_L$ and/or $K_S$ states.

[4] After much of this paper was written, we became aware of a theorem by Khalilin, as reported by Chiu and Sudarshan (Ref. [5]), which proves on very general grounds that, if the amplitude ratio whose square modulus is $R$ is a constant, the only possible value of $R$ is unity. While the theorem may be true, it requires $R$ to be constant at all times, and is invalidated by any variation of $R$. If such a variation were to occur at times outside the range of measurement, Khalilin's theorem would not affect our argument, which requires only that $K_L$ and $K_S$ be decoupled during the time interval under consideration. For further discussion, see Ref. [6].

[5] C.B. Chiu and E.C.G. Sudarshan, Phys. Rev. D 42, 3712 (1990). The assumption of TCP invariance made by these authors to deduce the constancy of $R$ is unnecessary (see Ref. [2]).


[7] If we set all $\theta_A$ equal to unity in $A_{jk}$, Eq. (6), we obtain the sum appearing in Eq. (10). Since we must initially have all $\theta_A = 1$, Eq. (10) expresses the condition $A_{jk}(t = 0) = \delta_{jk}$.


[11] This result is expected because we know, from the analysis of the CKM matrix, that for any unitary $3 \times 3$ matrix $D$ there is only one physically significant parameter which determines all possible $T$-noninvariant effects arising from $D$.