Ground-state energy of two-dimensional weakly coupled Hamiltonians

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Ground-state energy E of a weakly coupled Hamiltonian in two dimensions is analyzed in terms of the zeros of the inverse T matrix, and an explicit series expression for \( \ln(-E) \) is obtained.

I. INTRODUCTION

It is observed\(^1\) that a short-range attractive potential always produces a bound state in one or two dimensions. While this property has attracted considerable attention for one-dimensional potentials, leading to many general results,\(^2-7\) much less is known about the corresponding property in two dimensions. This may be due to the fact that the bound state is marginal in the case of two dimensions, and somewhat harder to analyze. Apart from the original result of Landau and Lifshitz,\(^1\) the only other results known for the two-dimensional problem are those obtained by Simon.\(^2\) For the two-dimensional Hamiltonian

\[
H = -\frac{1}{2} \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) + \lambda V(r),
\]

Simon showed that if

\[
\int |V(r)| \, d^2r < \infty,
\]

\[
\int (1 + r^2)^\epsilon |V(r)| \, d^2r < \infty, \quad \epsilon > 0
\]

then a bound state exists for all positive \( \lambda \), if and only if,

\[
\lambda \int V(r) \, d^2r < 0.
\]

He also showed that unlike the one-dimensional case, the ground-state energy \( E(\lambda) \) is never analytic at \( \lambda = 0 \) and that for \( \lambda \rightarrow 0 \),

\[
E(\lambda) \sim -\exp \left( \frac{\lambda}{2\pi} \int V(r) \, d^2r \right),
\]

which exhibits the nonanalyticity at \( \lambda = 0 \).

In the derivation of most of the results pertaining to bound states in weakly coupled potentials, the analysis has been generally in the coordinate space. Here, we prefer to use a momentum-space approach in terms of the T-matrix which allows us to analyze the bound states as poles of the T matrix. A particularly convenient form of the T matrix is the one provided by Noyes,\(^6\) which gives us a perturbative expansion for essentially the inverse of the T matrix, thus allowing a perturbative description of the poles of the T matrix and hence the bound-state energies. In our present note, the ground-state energy of Hamiltonian (1) is analyzed using the Noyes form\(^6\) of the T matrix. We find that for potentials satisfying conditions (2) and (3), the ground-state energy has an asymptotic series given by

\[
\ln[-E(\lambda)] = \frac{2\pi}{\lambda V_0} \left( 1 - \lambda \frac{C}{\pi} V_0 + \sum_{n=1}^{\infty} \lambda^n L_n \right), \quad \lambda \rightarrow 0,
\]

where

\[
V_0 = \int V(r) \, d^2r,
\]

\[
L_n = -\int \frac{d^2r_1 \cdots d^2r_{2n}}{(V_0)^{n}} V(r_1) P(r_{12}) \times [V(r_3) P(r_{34}) V(r_2) - V(r_3) V(r_4) P(r_{34})] \cdots \times [V(r_{2n-1}) P(r_{2n-1}) V(r_2) - V(r_{2n-1}) V(r_{2n}) P(r_{2n})] V(r_2),
\]

\[
P(r_{pq}) = \frac{1}{2} \pi^{-1} \ln \left( \frac{l_1}{l_p - l_q^2} \right),
\]

\( C \approx 0.5772 \) is the Euler constant, and for \( n = 1 \) there are no square brackets in the expression for \( L_1 \). This result is valid modulo exponentially vanishing terms and for \( V_0 \neq 0 \), although there will be a bound state\(^2\) for \( V_0 = 0 \) as well. However, we are unable to prove the convergence of the series. It is indeed a surprising result and is based on the fact that \( E(\lambda) \) is exponentially vanishing for \( \lambda \rightarrow 0 \). No such result is known for one-dimensional potentials.

II. RESULTS

We begin the T-matrix discussion with the Lippmann-Schwinger equation for the T matrix:

\[
\langle p | T | q \rangle = \langle p | \lambda V | q \rangle + \sum_k \frac{\langle p | \lambda V | k \rangle \langle k | T | q \rangle}{E - \frac{1}{2} k^2 + i\eta},
\]

where \( p, q, k \) are two-component vectors and where we put \( q^2 = 2E \) after carrying out all the integrations. For the analysis of the bound states it is
convenient to use the Noyes\textsuperscript{8} form of the equation which is obtained by writing
\[ f(k, q) = \frac{\langle k | V | q \rangle}{\lambda V_0} + \sum_q \left( \frac{k | \lambda V | k'}{\lambda V_0} - \frac{\langle k | \lambda V | q \rangle \langle q | \lambda V | k' \rangle}{\lambda V_0} \right) \times \frac{f(k', q)}{E - \frac{1}{2} k^2 + i\eta}, \tag{10} \]
where \( V_o = \langle q | V | q \rangle \). The bound-state energies of the Hamiltonian (1) then correspond to the zeros of the denominator of expression (9).

Now, expression (10) allows an iterative solution for \( f(k, q) \) so that one can write the denominator of expression (9) in the form
\[
D(E) = 1 - \lambda \int \frac{d^2 r_{1} d^2 r_{2}}{V_0} V(r_{1})G(|\vec{r}_{1} - \vec{r}_{2}|)V(r_{2})e^{i\mathbf{r}_{1} \cdot (\mathbf{r}_{2} - \mathbf{r}_{1})} \\
- \lambda^2 \int \frac{d^2 r_{1} d^2 r_{2} d^2 r_{3}}{(V_0)^2} V(r_{1})G(|\vec{r}_{1} - \vec{r}_{2}|)V(r_{2})G(|\vec{r}_{2} - \vec{r}_{3}|)V(r_{3}) - V(r_{2})V(r_{3})G(|\vec{r}_{3} - \vec{r}_{4}|)V(r_{4}) \\
\times e^{i\mathbf{r}_{1} \cdot (\mathbf{r}_{2} - \mathbf{r}_{1})} f(k, q) + \cdots , \tag{11} \]
where the square brackets repeat with appropriate indices for higher-order terms, and
\[
G(|\vec{r}_{1} - \vec{r}_{2}|) = \frac{1}{(2\pi)^2} \int d^2 k' \frac{e^{i\mathbf{k} \cdot (\mathbf{r}_{1} - \mathbf{r}_{2})}}{E - \frac{1}{2} k^2 + i\eta}. \tag{12} \]
We evaluate this function \( G \) for \( E < 0 \), by using the identity
\[
\frac{1}{E - \frac{1}{2} k^2 + i\eta} = - \int_0^\infty dt e^{-t(\mathbf{k}^2/2 - E + it)} \tag{13} \]
and carrying out the \( k \) integrations. One then obtains
\[
G(|\vec{r}_{1} - \vec{r}_{2}|) = - \frac{1}{2\pi} \int_0^\infty \frac{dt}{t} e^{i\mathbf{r}_{1} \cdot (\mathbf{r}_{2} - \mathbf{r}_{1})^2 / 2t}, \tag{14} \]
which on integration\textsuperscript{7} yields
\[
G(|\vec{r}_{1} - \vec{r}_{2}|) = \frac{1}{\pi} \left\{ i \left( \ln x \right) \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \right\} \\
- \sum_{\eta=0}^{\infty} \frac{x^n}{(n+1)!} \psi(n+1), \tag{15} \]
where
\[
x = \frac{1}{2} \left| E \right| \left( \vec{r}_{1} - \vec{r}_{2} \right)^2, \tag{16} \]
\[
\psi(x) = \frac{1}{\Gamma(x)} \frac{d}{dx} \Gamma(x). \tag{17} \]
We now note the important well-known property\textsuperscript{1,2} that \( E(\lambda) \), and also \( q \) are exponentially small for \( \lambda \rightarrow 0^{-} \), i.e., \( E(\lambda) \approx \exp(1/\lambda) \), as seen in (4). Thus we have
\[
G(|\vec{r}_{1} - \vec{r}_{2}|) = \frac{1}{2\pi} \left( \left( \ln x \right) + 2C \right) + O(e^{1/\lambda}) \tag{17} \]
for \( \lambda \rightarrow 0^{-} \), where \( C = e^{-\psi(1)} \) is the Euler constant.
We also note that the \( r \)-independent part of the \( G \) function does not contribute to the square brackets in (11). Therefore, for potentials satisfying conditions (2) and (3), one has
\[
D(E) = 1 - \frac{\lambda}{2\pi} \left\{ \left( \ln \frac{|E|}{2} \right) + 2C \right\} - \lambda \int \frac{d^2 r_{1} d^2 r_{2}}{V_0} V(r_{1})P(r_{12})V(r_{2}) \\
- \lambda^2 \int \frac{d^2 r_{1} d^2 r_{2} d^2 r_{3} d^2 r_{4}}{(V_0)^2} V(r_{1})P(r_{12})V(r_{23}) - V(r_{2})V(r_{3})P(r_{34})V(r_{4}) - \cdots + O(e^{1/\lambda}) \tag{18} \]
for \( \lambda \rightarrow 0^{-} \), where \( V_0 \) and \( P(r_{ij}) \) are defined in (6). For the energy of the bound state we have the condition
\[
D(E) = 0, \tag{19} \]
which immediately leads to the result (5). We only add that the same result could presumably be obtained from other approaches, e.g., that of Simon.\textsuperscript{2}
III. DISCUSSION

We have evaluated the first few terms in expression (5) for a particle in a two-dimensional well of depth \( \lambda \) and radius \( a \). The angular integrations are carried out by noting that

\[
\int_0^{2\pi} d\theta \ln (r_1^2 + r_2^2 - 2r_1r_2 \cos \theta) = 4\pi [(\ln r_1)\Theta(r_1 - r_2) + (\ln r_2)\Theta(r_2 - r_1)],
\]

where \( \Theta(r) \) is the step function. Carrying out the remaining integrations and using \( V_0 = \pi a^2 \), one gets

\[
\ln(-E) = \left(2/\lambda a^2\right)[1 - \lambda\alpha^2 + \frac{1}{3}\lambda^3\alpha^4 - \ln\alpha^4] \\
- \frac{1}{48}\lambda^5\alpha^6 + \frac{1}{384}\lambda^7\alpha^8 - \ldots \\
+ O(e^{1/\lambda}),
\]

for \( \lambda \to 0 \). This expression is in agreement with the iterative solution obtained from the exact eigenvalue condition for the well,

\[
\frac{J_{l}^\prime[(2E - \lambda)^{1/2}a]}{J_{l}[(2E - \lambda)^{1/2}a]} = \frac{(2|E|)^{1/2}}{2(E - \lambda)^{1/2}} \frac{H_{l+1}^{(1)}([2(E - \lambda)^{1/2}a])}{H_{l+1}^{(1)}([2E - \lambda]^{1/2}a)},
\]

where \( J_{l} \) and \( H_{l}^{(1)} \) are the Bessel and Hankel functions and \( J_{l+1}^{\prime} \) and \( H_{l+1}^{(1)} \) are their derivatives.

It might generally be expected that the weak-coupling bound-state problem in two dimensions would be more difficult than the corresponding problem in one dimension, and indeed the problem is complicated by nonanalyticity at \( \lambda = 0 \). This complication results in an exponentially vanishing expression (4) for \( E(\lambda) \) as \( \lambda \to 0 \). Interestingly enough it is this special property which allows us to obtain in two dimensions, an explicit series for \( \ln(-E) \), modulo exponentially vanishing terms.

Solutions in two dimensions are of importance because they may provide a good simulation of the more complicated solutions in 3 dimensions. It has been emphasized by Van Vleck that the radial solutions in two dimensions for angular momentum quantum number \( l \), correspond to radial solutions \( r^{1/2}R_{l+1/2} \) in three dimensions for angular-momentum quantum number \( l - \frac{1}{2} \). This being the case, our solution (5) corresponds to locating the three dimensional Regge pole for weakly coupled, attractive potentials at \( l = -\frac{1}{2} \), for energies given by expression (5). This result might appear surprising in view of the fact that weakly coupled potentials do not necessarily produce bound states in three dimensions, for physical \( l \) values. However, it appears more reasonable if one notes that for the unphysical value of \( l = -\frac{1}{2} \), the angular momentum term is attractive and that for \( l(l+1) < -\frac{1}{4} \), the particle "falls" to the center.\(^{10}\)

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10. L. D. Landau and E. M. Lifshitz, Quantum Mechanics (Pergamon, New York, 1958), p. 120.