Analysis of Faulted Power Systems in Three Phase Coordinates—A Generic Approach

Rajeev Kumar Gajbhiye, Student Member, IEEE, Pushpa Kulkarni, Student Member, IEEE, S. A. Soman

Abstract—Analysis of faulted power systems in 3-phase co-ordinates is desirable when dealing with unbalanced networks and complex faults. Thevenin’s model in 3-phase domain is well-known for shunt faults. However, lack of generic circuit model for series and simultaneous faults is an impediment, as it sacrifices both simplicity and computational efficiency. In this paper, we extend this approach to arbitrarily complex faults. Case studies with complex faults on two example systems demonstrate the claims made in the paper.

Index Terms—Fault Circuit Analysis, Phase Coordinates, Thevenin’s approach

I. INTRODUCTION

Analysis of standard shunt faults in sequence domain is based upon construction of Thevenin’s equivalent circuit in sequence domain. However, analysis of shunt, series and simultaneous faults in sequence domain is cumbersome [1]. Further, sequence domain analysis requires a balanced network assumption. More precisely, network elements should have circulant symmetry. Unfortunately, distribution systems as well as transmission systems with untransposed lines are unbalanced. This motivates analysis in 3-phase co-ordinates which does not require balanced network assumption [2], [3]. Even in the 3-phase domain, Thevenin’s equivalent circuit representation is available for standard shunt faults [4]–[6]. We now briefly describe 3-phase Thevenin’s model for the analysis of shunt faults

Let $Y_f$ represent a $3 \times 3$ shunt fault admittance matrix for a fault at the $i^{th}$ bus of a network and 3-phase pre-fault admittance matrix model be given by the following equation

$$I_{inj}^{old} = Y_{bus}^{old}V_{bus}^{old}$$

Then, under the assumption that inverse of the matrix $(I_3 + Z_{bus}^{old}(i,i) Y_f)$ exists, where $I_3$ is an identity matrix of size $3 \times 3$, Thevenin’s equivalent circuit in 3-phase co-ordinates is given by the following equation:

$$V_i^{old} = Z_{th}^{sh}I_f^{sh} + V_f^{sh}$$

$$I_f^{sh} = Y_f^{sh}V_f^{sh}$$

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R. K. Gajbhiye, P. Kulkarni and S. A. Soman are with the Indian Institute of Technology, Bombay 400076, India (e-mail: rajeev81@ee.iitb.ac.in; kpushpa@ee.iitb.ac.in; soman@ee.iitb.ac.in).

Superscript $old$ and $new$ refer to pre and post fault network quantities respectively. This result can be proved by application of compensation theorem [5]. An alternative proof using Woodbury’s generalization of Sherman- Morrison formula has been presented in [6]. Fig 1 provides the visualization of this model. The representation is simple and straightforward. This simplifies the analysis of standard shunt faults.

However, series faults do not have such simpler version of analysis. Typically, analysis of complex faults involves specifying additional constraint equations like $I_a = 0$ for phase $a$ open circuit, $V_a = 0$ for bolted short circuit on phase $a$ etc. Elimination of such constraints leads to a reduced system of equations [7]. An alternative approach is to approximate bolted shunt and series faults by approximate small and large impedances and modify the admittance matrix [8]. Either way, we obtain a modified network admittance matrix formulation. Solution of the resulting system of equations provides the post fault system voltages. Such an approach is computationally inefficient, as a new matrix refactorization is required for a fault. Yet another strategy is to use hybrid approach [9], [10] where in mixed phase and sequence coordinate representation is used. This leads to computational efficiency. However, in our opinion, such an approach is also complex. In [11], Tinney has presented a compensation approach, wherein opening of a branch is modeled by the addition of a parallel current source of appropriate value. For the case of mutually coupled branches, with one branch being opened, compensating current

![Fig. 1. Thevenin’s equivalent circuit for shunt fault](image-url)
source has to be attached in parallel to both the branches. The advantage of this approach is that original \( Y_{bus} \) need not to be modified and hence no need of refactorization. In [12], the idea of compensated Thevenin’s impedance is developed to accommodate line outage in fault analysis. The key idea is to express network changes as low-order matrix product of the form \( BRC^T \). Fault analysis of simultaneous shunt faults using Thevenin’s model is also discussed in [13].

To conclude, a considerable amount of literature exists on fault analysis, both in sequence and 3-phase domain. However, its study reveals lack of an integrated yet simplified treatment. One of the goals of this paper is to provide such an unified treatment that includes generalization of Thevenin’s model to its study reveals lack of an integrated yet simplified treatment. Schematically, such a fault can be represented by fig 2(a).

2) A fault which alters both \( 3 \times 3 \) block diagonal entries and off diagonal entries in the 3-phase \( Y_{bus} \). Such faults will be referred as \( series \) faults, e.g. open circuit fault due to opening of any phase conductor of a transmission line belongs to this class. Schematically, such a fault can be represented by fig 2(b). For phase open circuit on a line, \( \Delta Y_{sr} \) is the alteration in the line admittance.

3) A fault which also alters the injection of current / voltage in the system. Open circuit of a generator phase belongs to this class. It results in alteration in current injection and shunt admittances. Schematically, such a fault can be represented by fig 2(c).

A fault which involves multiple instances of either the same or different fault types (1–3) will be referred as \( simultaneous \) fault. In what follows, we show that faults of type 1 and 2 can be effectively handled through Thevenin’s equivalent circuit in 3-phase coordinates. Faults of type 3 in addition, require application of superposition theorem.

III. THEVENIN’S EQUIVALENT IN PHASE COORDINATES

A. Analysis of Series Faults

Series fault refers to fault between busses, like opening of phase a conductor. Such faults are common in distribution systems. Let a series fault be created in a branch connected between nodes \( i \) and \( j \). Correspondingly, let the \( 3 \times 3 \) primitive admittance of branch change from \( Y_{old} \) to \( Y_{new} \). The resulting change will modify the four block entries namely \((i, i), (i, j), (j, i)\) and \((j, j)\) in \( Y_{bus} \). Let,

\[
\Delta Y_{sr} = Y_{new} - Y_{old}
\]

The modification of block diagonal entries will involve addition of \( \Delta Y_{sr} \) where as the modification of block non-diagonal entries will involve subtraction of \( \Delta Y_{sr} \). So the modified block entries in new admittance matrix will be given by,

\[
\begin{align*}
Y_{bus}^{new}(i, i) &= Y_{bus}^{old}(i, i) + \Delta Y_{sr} \\
Y_{bus}^{new}(i, j) &= Y_{bus}^{old}(i, j) - \Delta Y_{sr} \\
Y_{bus}^{new}(j, i) &= Y_{bus}^{old}(j, i) - \Delta Y_{sr} \\
Y_{bus}^{new}(j, j) &= Y_{bus}^{old}(j, j) + \Delta Y_{sr}
\end{align*}
\]

This modification can be represented as the rank-6 update,

\[
Y_{bus}^{new} = Y_{bus}^{old} + [E_i E_j \bar{\hat{Y}}_{sr} [E_i E_j]^T
\]
where,

\[ Y_{fsr} = \begin{bmatrix} \Delta Y_{fr} & -\Delta Y_{fr} \\ -\Delta Y_{fr} & \Delta Y_{fr} \end{bmatrix} \tag{6} \]

\( E_i \) is a block vector such that,

\[ E_i(j) = O_3 \quad i \neq j \]
\[ = I_3 \quad i = j \]

with \( O_3 \) being zero matrix of size 3 \( \times \) 3 and \( I_3 \) is identity matrix of the same size.

The key result of this work is stated by the following theorem.

**Theorem 1 (Series Fault).** For a series fault created in a branch between bus \( i \) and \( j \), post-fault voltages and fault currents at the faulted buses are given by the following relationships,

\[ \begin{bmatrix} V_{new} \\ I_f \end{bmatrix} = \begin{bmatrix} I_0 + Z_{th} Y_{fsr} \\ \end{bmatrix}^{-1} \begin{bmatrix} V_{old} \\ I_f \end{bmatrix} \tag{7} \]

\[ I_f^t = Y_{fnew} \left( V_{fnew} - V_{fnew} \right) \tag{8} \]

where, \( \hat{Y}_{fsr} \) is given by (6) and \( Z_{th} \) is given by (9).

Equation 7 describes the Thevenin’s equivalent circuit for the series fault with \( Z_{th} \) given by equation (9). Fig 3 visualizes this equation. The Thevenin’s equivalent circuit consists of two 3-phase impedances \( Z_{bus}(i,i) \) and \( Z_{bus}(j,j) \) mutually coupled with each other by \( Z_{old}(i,j) \) and \( Z_{old}(j,i) \). Current in the faulted line \( I_f \) is given by eqn 8. The current \( I_f \) is a fictitious current associated with the sub-network depicting the change due to the fault.

**B. Simultaneous Faults**

We now consider the case of simultaneous fault where one or more busses may be involved in more than one fault. The simultaneous fault is composed of shunt and series faults. Let the set of busses involved in the fault be \( S = \{i_1, i_2, \ldots, i_m \} \). Let us assume that \( m_{sh} \) number of shunt faults and \( m_{sr} \) number of series faults occur on busses that belong to the set \( S \). Let the fault admittance \( Y_{jg}^{th} \) represent a shunt admittance to be connected on bus \( k \) and \( Y_{jg}^{sr} \) represents series fault admittance required between bus \( k \) and \( l \) to capture the post-fault changes in the network. \( Y_{bus}^{new} \) can be obtained from \( Y_{bus}^{old} \) by the following sequence of steps:

1. Set \( \Delta Y \) to \( n \times n \) zero block matrix.
2. Identify subset \( S_{sh} \) of busses in set \( S \) on which shunt fault occurs. Similarly, identify subset \( S_{sr} \) of busses from set \( S \) involved in series fault. Note that \( S = S_{sh} \cup S_{sr} \) and \( S_{sh} \cap S_{sr} \) need not be empty.
3. For all the shunt faults \( Y_{fs}^{th}, i_s \in S_{sh} \), add \( Y_{fs}^{th} \) to \( \Delta Y(i_s, i_s) \).
4. For all the series faults \( Y_{fs}^{sr}, i_s, i_t \in S_{sr} \), add the said admittance to the diagonal entries \( \Delta Y(i_s, i_t) \). Also subtract the same admittance from \( \Delta Y(i_s, i_t) \).

Finally,

\[ Y_{bus}^{new} = Y_{bus}^{old} + \Delta Y \tag{10} \]

In matrix \( \Delta Y \), all the entries except those belonging to faulted nodes, are zero (or more precisely \( O_3 \)). Eqn 10 can also be expressed as rank-3m update of \( Y_{bus}^{old} \).

\[ Y_{bus}^{new} = Y_{bus}^{old} \cdot \begin{bmatrix} [E_{i_1:i_m}] & Y_{fs}^{sim} & [E_{i_1:i_m}] \end{bmatrix}^T \tag{11} \]

where, \( [E_{i_1:i_m}] = \begin{bmatrix} E_{i_1}, E_{i_2}, \ldots, E_{i_m} \end{bmatrix} \). An entry \( (k, l) \) in \( Y_{fs}^{sim} \) is given by \( \Delta Y(k, l) \). \( k, l \in \{1, 2, \ldots, m \} \), i.e., \( Y_{fs}^{sim} \) is a compressed representation of \( \Delta Y \), in which all zero rows and columns of \( \Delta Y \) have been eliminated.

Proceeding in a similar fashion to the case of series fault, one can verify that post fault bus voltages at faulted busses can be calculated using following equation

\[ V_f = \begin{bmatrix} V_{new}(i_1) \\ V_{new}(i_2) \\ \vdots \\ V_{new}(i_m) \end{bmatrix}^T \]
\[ = (I_{3m} + Z_{th}^T Y_{fs}^{sim} I_{3m}^{-1}) V_{sim}^T \tag{12} \]

where, resultant Thevenin’s voltage source vector and impedance matrix are given by the following relationships:

\[ V_{sim}^T = \begin{bmatrix} V_{old}(i_1) \\ V_{old}(i_2) \\ \vdots \\ V_{old}(i_m) \end{bmatrix}^T \tag{13} \]
\[ Z_{th} = \begin{bmatrix} Z_{bus}(i_1, i_1) & Z_{bus}(i_1, i_2) & \cdots & Z_{bus}(i_1, i_m) \\ Z_{bus}(i_2, i_1) & Z_{bus}(i_2, i_2) & \cdots & Z_{bus}(i_2, i_m) \\ \vdots & \vdots & \ddots & \vdots \\ Z_{bus}(i_m, i_1) & Z_{bus}(i_m, i_2) & \cdots & Z_{bus}(i_m, i_m) \end{bmatrix} \tag{14} \]
Fig 4 illustrates the scheme to analyze simultaneous fault using Thevenin’s equivalent circuit. Figure 5 illustrates Thevenin’s equivalent circuit for simultaneous fault at four buses in a system.

C. Calculation of Post Fault Voltages

It can be shown by the application of superposition theorem that, post fault voltages are given by the following equations.

1) Shunt Fault:

$$V_{new}^{old} = V_{old}^{old} + Z_{bus}^{old} (E_i - I_f^{sh})$$  \hspace{1cm} (15)

2) Series Fault: Fictitious current in fig 3 represents the compensation current required to model the post fault system. Hence, by the application of superposition theorem

$$V_{new}^{old} = V_{old}^{old} + Z_{bus}^{old} [E_i E_j] \begin{bmatrix} I_f^{sr} \\ -I_f^{sr} \end{bmatrix}$$  \hspace{1cm} (16)

where, $i$ and $j$ are the faulty buses, and

$$\begin{bmatrix} I_f^{sr} \\ -I_f^{sr} \end{bmatrix} = \hat{Y}_f^{sr} \begin{bmatrix} V_{new}^{old} \\ V_{new}^{old} \end{bmatrix}$$

$I_f^{sr}$ is as shown in fig 3.

3) Simultaneous Fault: Let the fictitious current vector associated with sub-network depicting the change be represented as

$$I_f^{fict} = \begin{bmatrix} I_f^{fict}^{T} & I_f^{fict}^{T} & \cdots & I_f^{fict}^{T} \end{bmatrix}^T$$

$I_f^{fict}$ can be calculated as follows:

$$I_f^{fict} = \hat{Y}_f^{sim} \begin{bmatrix} V_{new}^{old}(i_1) \\ V_{new}^{old}(i_2) \\ \vdots \\ V_{new}^{old}(i_m) \end{bmatrix}$$  \hspace{1cm} (17)

Now, post fault voltages can now be calculated by applying superposition theorem.

$$V_{new}^{old} = V_{old}^{old} + Z_{bus}^{old} [E_i E_i \cdots E_i_m] [-I_f^{fict}]$$ \hspace{1cm} (18)

Remark 1. Knowing the postfault voltages, postfault current can be easily computed for each component in the system from their primitive 3-phase admittance matrix.

IV. TEST CASES

In this section, we consider faults whose analysis in sequence component is cumbersome and complex. Further, we show how their analysis in 3-phase coordinates is simplified.

A. Analysis of Open Conductor Falling on Ground

The single line diagram of the system [1] along with sequence data for base values of 30MVA, 34.5KV is shown in figure 6. The positive and negative sequence impedances of load are 1.0∠25.84° p.u, 0.60∠29° p.u. respectively. Load voltage is kept at 1.0 p.u.

Consider the case of a simultaneous fault. Conductor in phase $a$ of transmission line between bus $G$ and bus $H$ breaks and it gets grounded on the load side. Now we apply the Thevenin’s equivalent method, for simultaneous fault (section III-B). Block diagram of Thevenin’s equivalent circuit is represented in figure 7. Here, Thevenin’s impedance $Z_{th}$ and fault admittance $Y_f^{sim}$ are block matrices of size $2 \times 2$. Entries of $Y_f^{sim}$ are given as follows:

$$Y_f^{sim} = \begin{bmatrix} \Delta Y_f^{sr} & -\Delta Y_f^{sr} \\ -\Delta Y_f^{sr} & \Delta Y_f^{sr} + Y_f^{sh} \end{bmatrix}$$

where, $\Delta Y_f^{sr}$ is the series admittance (see equation 4) representing the change due to opening of phase $a$ of transmission.
line between busses G and H. $Y_{ph}^f$ is shunt fault admittance representing single line to ground (SLG) fault on phase $a$ of bus H. Though $Y_{ph}^f$ cannot be represented for bolted fault, it is constructed by assuming very small value of ground impedance ($10^{-6}$ p.u.). Thus, fault bus voltages is calculated using eqn 12. This leads to the following results:

$$
\begin{bmatrix}
V_{G}^{new} \\
V_{H}^{new}
\end{bmatrix}
= (I_{f} + Z_{th}^s Y_{sh}^{sim})^{-1} \begin{bmatrix}
V_{G}^{old} \\
V_{H}^{old}
\end{bmatrix} \tag{19}
$$

$$
I_{f} = \begin{bmatrix}
1.26199, \angle -58.809^\circ \\
1.01855, \angle 5.175^\circ \\
1.68279, \angle -91.727^\circ 
\end{bmatrix} \tag{20}
$$

The result is consistent with sequence component analysis [1].

B. Simultaneous fault on system with mutually coupled lines

In this example, taken from [8], we analyze 345 kV double circuit transmission line for simultaneous fault on it. T-type equivalent of the 100 mi double circuit 345 kV transmission line with equivalent impedances of the receiving and the sending end systems is reproduced here in fig 8. In [14], a procedure to calculate $Y_{bus}^{old}$ in sequence domain considering the mutual couplings has been presented. Similar steps are followed in 3-phase domain to obtain 3-phase $Y_{bus}$. The simultaneous fault considered for this system is a SLG fault on phase $a$ of bus 2 and open phase fault on phase $b$ of generator on bus 4. Open circuit fault on a generator bus is a shunt fault. It has a peculiar nature that it not only modifies the $Y_{bus}$ but it also alters the current injection. Hence, this fault is combination of type 1 and type 3 faults. The effect of change in current injection on post fault system is analyzed by using superposition theorem. Fault analysis is done as follows:

- $Z_{th}^{sim}$ is formed by picking proper entries from $Z_{bus}^{old}$ corresponding to buses involved in fault. Figure 9 shows Thevenin’s equivalent circuit for simultaneous fault. Shunt admittance $Y_{ph}^f$ is connected at bus two and open circuit of generator is shown by a current injection $\Delta I$ with shunt admittance $\Delta Y_{ph}^f$.

- $\Delta I$ is calculated as

$$
\Delta I = I_{gen2}^{new} - I_{gen2}^{old} = Y_{gen2}^{new} E_{gen2} - Y_{gen2}^{old} E_{gen2} = \Delta Y_{ph}^f E_{gen2}
$$

Define for the faulted busses,

$$
\Delta I_{inj} = \begin{bmatrix}
O_3 \\
\Delta I
\end{bmatrix}
$$

- Using eqn (12), post fault voltages at faulted busses are calculated as follows:

$$
\begin{bmatrix}
V_{2}^{new} \\
V_{4}^{new}
\end{bmatrix}
= (I_{f} + Z_{th}^{sim} Y_{f}^{sim})^{-1} \begin{bmatrix}
V_{2}^{old} \\
V_{4}^{old}
\end{bmatrix} \times \left( \begin{bmatrix}
Y_{2}^{old} \\
Y_{4}^{old}
\end{bmatrix} + Z_{th}^{sim} \Delta I_{inj} \right) \tag{21}
$$

The additional term $Z_{th}^{sim} \Delta I_{inj}$ in above equation is a consequence of superposition effect of $\Delta I$ current injection.

- Fictitious current vector (see fig 9) is calculated by using the following equation

$$
I_{f}^{fict} = \begin{bmatrix}
\Delta Y_{ph}^f V_{2}^{new} \\
\Delta Y_{ph}^f V_{4}^{new} - \Delta I
\end{bmatrix}
$$

The post fault voltages are calculated by eqn (18). These are tabulated in table I. They are consistent with results in [8], which validates the proposed approach.

V. CONCLUSIONS

Three phase analysis is suitable for fault analysis of unbalanced systems and complex faults. In this paper, we have proposed generic Thevenin’s circuit representation for shunt, series and simultaneous faults. It’s advantage is conceptual simplicity and computational efficiency as repeated matrix factorization is not required. Two test systems with complex faults were analyzed to validate the approach.

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REFERENCES


APPENDIX I  
PROOF OF THEOREM 1

The Woodbury’s formula [15] for calculating inverse of a rank-\(k\) updated matrix \(A\) is given by,

\[
(A + UV^T)^{-1} = A^{-1} - A^{-1}U (I_k + VT A^{-1} U)^{-1} V T A^{-1}
\]  

(22)

where, \(A\) is \(n \times n\) matrix, \(U\) and \(V\) are \(n \times k\) matrices and \(I_k\) is \(k \times k\) identity matrix. It is assumed that necessary inverse in eqn (22) exists.

For the case of series fault, from equation (5), we set

\[
Z_{bus}^{new} = \left[ Y_{bus}^{old} + [E_i E_j] \hat{Y}_{sf} [E_i E_j]^T \right]^{-1}
\]

where, \(\hat{Y}_{sf}\) is defined in the eqn (6).

Applying Woodbury’s formula (eqn 22) with \(A = Y_{bus}^{old}\), \(U = [E_i E_j]\) and \(V = [E_i E_j]\)

\[
Z_{bus}^{new} = Z_{bus}^{old} - Z_{bus}^{old} [E_i E_j] \hat{Y}_{sf} \left( I_6 + [E_i E_j]^T Z_{bus}^{old} [E_i E_j] \right)^{-1} [E_i E_j]^T Z_{bus}^{old} [E_i E_j] \hat{Y}_{sf}
\]

\[
= Z_{bus}^{old} - \left[ Z_{bus}^{old} (i, i) Z_{bus}^{old} (j, j) \right] \hat{Y}_{sf}
\]

\[
\left( I_6 + \left[ Z_{bus}^{old} (i, i) Z_{bus}^{old} (j, j) \right] \hat{Y}_{sf} \right)^{-1} \left[ Z_{bus}^{old} (i, i) Z_{bus}^{old} (j, j) \right] \hat{Y}_{sf}
\]

(23)

Post-multiplying \(f_{old}^i\) and sampling \(i^{th}\) and \(j^{th}\) row and substituting \(Z_{bus}^{old} (i, i) I_{inj}^{old} = V_i^{old}\) and \(Z_{bus}^{old} (j, i) I_{inj}^{old} = V_j^{old}\),

\[
\begin{align*}
V_i^{new} & = V_i^{old} - \left[ Z_{bus}^{old} (i, i) Z_{bus}^{old} (i, j) \right] \hat{Y}_{sf} \\
V_j^{new} & = V_j^{old} - \left[ Z_{bus}^{old} (j, i) Z_{bus}^{old} (j, j) \right] \hat{Y}_{sf}
\end{align*}
\]

\[
\left( I_6 + \left[ Z_{bus}^{old} (i, i) Z_{bus}^{old} (j, j) \right] \hat{Y}_{sf} \right)^{-1} \left[ V_i^{old} V_j^{old} \right]
\]

(24)

Let,

\[
\begin{align*}
V_i^{old T} & V_j^{old T} = V_{th} \\
V_i^{new T} & V_j^{new T} = V_f \\
Z_{bus}^{old} (i, i) & Z_{bus}^{old} (j, i) \\
Z_{bus}^{old} (j, i) & Z_{bus}^{old} (j, j)
\end{align*}
\]

Then we have

\[
V_f = V_{th} - Z_{th} \hat{Y}_{sf} \left( I_6 + Z_{th} \hat{Y}_{sf} \right)^{-1} V_{th}
\]

\[
= \left( \left( I_6 + Z_{th} \hat{Y}_{sf} \right)^{-1} - Z_{th} \hat{Y}_{sf} \left( I_6 + Z_{th} \hat{Y}_{sf} \right)^{-1} \right) V_{th}
\]

\[
\Rightarrow V_f \left( I_6 + Z_{th} \hat{Y}_{sf} \right)^{-1} V_{th}
\]

(28)

Rajeev Kumar Gajbhiye is currently working towards Ph.D. degree in Department of Electrical Engineering at Indian Institute of Technology, Bombay, India. His research interests include power system analysis and object oriented analysis.

Pushpa Kulkarni is currently pursuing M.Tech degree in Indian Institute of Technology, Bombay, India.