Preservation of 2-D Signal Symmetries in Quincunx Filter-Banks

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Abstract—An important problem in the analysis of symmetric extension methods is to determine the conditions under which signal symmetries are preserved by the filtering (convolution) and downsampling operations. In this letter, we provide a complete characterization of four-fold two-dimensional signal symmetries viz. quadrantal symmetry, diagonal symmetry, and 90° rotational symmetry. We then consider Quincunx filter-banks and determine the conditions under which the four-fold signal symmetries are preserved by the filtering and downsampling operations.

Index Terms—Quincunx filter banks, two-dimensional (2-D) signal symmetries.

I. INTRODUCTION

In the case of one-dimensional (1-D) multirate filter-banks, preservation of signal symmetry by the filtering (convolution) and downsampling operations [7] is one of the important parts in the analysis of symmetric signal extension schemes [3]–[5]. For two-dimensional (2-D) signals, a larger variety of symmetries are possible (than the 1-D case), like quadrantal symmetry, centro symmetry, diagonal symmetry, four-fold 90° rotational symmetry, octagonal symmetry, etc. [6], [9] (note that in the 2-D case, linear-phase implies only centro-symmetry). Recently, the authors of [2] presented a symmetric signal extension scheme for the 2-D two-channel Quincunx filter-banks. The following was shown in [2] for Quincunx filter-banks:

1) When the input signal is quadrantal symmetric, and the filter impulse-response also has quadrantal symmetry, then the filtered output is quadrantal symmetric (see [2, Lemma 2]).

2) A quadrantal symmetric signal becomes a diagonally symmetric signal after downsampling (see [2, Theorem 1]).

Thus, we can say that, for quadrantal symmetric input signals, the filtering and Quincunx downsampling operations preserve the symmetry. By “preservation of symmetry,” we mean that the subband signals are symmetric (not necessarily the same type of symmetry as the input). We note that [2] only considers the case when the input signal has quadrantal symmetry. In this letter, we generalize the above results for all the four-fold symmetries (quadrantal, diagonal, and 90° rotational), i.e., we show that the symmetry preservation property of Quincunx filter-banks holds when the input signal has any of the four-fold signal symmetries.

The rest of this letter is organized as follows: In Section II, we give a characterization of the four-fold 2-D signal symmetries. In Section III, we analyze the effect of filtering on the signal symmetry, and in Section IV, we analyze the behavior of the four-fold signal symmetries under Quincunx downsampling.

Notation: Boldfaced lowercase letters are used to represent vectors, and boldfaced uppercase letters are used for matrices. The lattice generated by a sampled matrix M is denoted as LAT(M). Throughout this letter, we will use \( Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \) as the generating matrix for the Quincunx lattice. Fig. 1(a) shows the Quincunx filter-bank.

II. CHARACTERIZATION OF 2-D SIGNAL SYMMETRIES

We generalize the characterization of 2-D symmetries from [6] and [9], to include the case where the center of symmetry is not the origin. A 2-D signal is said to be symmetric if \( x[Tn + b] = x[n] \) (identity-symmetry) or \( x[Tn + b] = -x[n] \) (antisymmetry). Here \( b \) is a \( 2 \times 1 \) vector, and \( T \) is a non-singular matrix. The most commonly used \( T \) matrices (and the ones we use in this letter) are

\[
T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\
T_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad T_5 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T_6 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

(Note that \( T_1T_2 = T_3T_4 = T_5^2 = -I \), and \( T_6 = T_3^2 \)).

We say that a symmetry is \( k \)-fold if there are \( k \) “identical regions.” In this letter, we only consider four-fold symmetries, i.e., quadrantal symmetry, diagonal symmetry, and 90° rotational symmetry. Below, we define these four-fold symmetries and discuss the various possibilities in each of these symmetry-types.

For convenience of notation, we define constant matrices \( A_i \), \( i = 1, \ldots, 6 \), as \( A_i = (1/2)(I - T_i) \), for \( i = 1, 2 \), and \( A_i = I - T_i \), for \( i = 3, 4, 5, 6 \), where \( I \) is the identity matrix.

A. Quadrantal Symmetry

Quadrantal symmetry, with center of symmetry \( c = [c_1 \ c_2]^T \), can be defined as follows: \( x[n] = x[T_1n + 2A_1c] = x[T_2n + 2A_2c] = x[T_3T_4n + 2c] \).

There are four different types for the location of the center of symmetry, with \( c_1 \) and \( c_2 \) each independently taking the value of a “full integer (F),” i.e., \( Z \) (where \( Z \) denotes the set of integers) or a “half integer (H),” i.e., \( (1/2)Z_{odd} \) (where \( Z_{odd} \) denotes the set of odd integers). We abbreviate these four cases as FF (\( c_1 \) and \( c_2 \) are both F), FH (\( c_1 \) is F, \( c_2 \) is H), HF (\( c_1 \) is H, \( c_2 \) is F),
and HH ($c_1$, $c_2$ are both H). Fig. 2 shows these four cases of quadrantally symmetric.

The above symmetries use identity-symmetry. When considering antisymmetry, we can have antisymmetry independently for the $T_1$ and $T_2$ operations, i.e., we can have

$$x[n] = \gamma_1 x[T_1n + 2A_1c] = \gamma_2 x[T_2n + 2A_2c] = \gamma_1 \gamma_2 x[T_1T_2n + 2c].$$

In z-domain

$$X(z) = \gamma_1 z^{-2A_1c} X(zT_1) = \gamma_2 z^{-2A_2c} X(zT_2) = \gamma_1 \gamma_2 z^{-2c} X(zT_1T_2).$$

For notation, we have used the same subscript for $\gamma$ as its associated $T$-operation. $\gamma_1$ and $\gamma_2$ can each independently be $+1$ (symmetry) or $-1$ (antisymmetry), thus giving four possibilities: SS, SA, AS, and AA. Fig. 1(b)-(1d) shows the three antisymmetry cases when the center of symmetry is of type FH [the quadrant case is shown in Fig. 2(b)]. So, to summarize, for quadrantally symmetric, there are 16 possibilities as tabulated in Table I(a).

### B. Diagonal Symmetry

Diagonal symmetry, with center of symmetry $c = [c_1 \ c_2]^T$, can be characterized as follows:

$$x[n] = x[T_{3n} + A_3c] = x[T_{4n} + A_4c] = x[T_3T_4n + 2c].$$

Fig. 3(a) shows $x[n]$ (which is diagonally symmetric) and $x[T_{3n}]$, $x[T_{4n}]$, and $x[T_3T_4n]$ to illustrate that the above characterization holds. In the case of diagonal symmetry, due to the nature of the symmetry, the center of symmetry can only be of type FF or HH. The center of symmetry in the signal in Fig. 3(a) is of type FF. Also, similar to the case of quadrantally symmetric, a symmetry or antisymmetry can be associated independently with $T_3$ and $T_4$ operations. This can be written as

$$x[n] = \gamma_3 x[T_{3n} + A_3c] = \gamma_4 x[T_{4n} + A_4c] = \gamma_3 \gamma_4 x[T_3T_4n + 2c].$$

In z-domain

$$X(z) = \gamma_3 z^{-A_3c} X(zT_3) = \gamma_4 z^{-A_4c} X(zT_4) = \gamma_3 \gamma_4 z^{2c} X(zT_3T_4).$$

$\gamma_3$ and $\gamma_4$ can each independently be $+1$ or $-1$, thus giving four possibilities of symmetry/antisymmetry (just like the quadrant case). Combining this with the two types of the center of symmetry, the different types of diagonal symmetries can be summarized as in Table I(b).

### C. 90° Rotational Symmetry

90° rotational symmetry, with center of symmetry $c = [c_1 \ c_2]^T$, can be characterized as follows:

$$x[n] = x[T_{3n} + A_{3c}] = x[T_3n + 2c] = x[T_{4n} + A_{4c}].$$

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**Table I**

<table>
<thead>
<tr>
<th>SS ($\gamma_1=1, \gamma_2=1$)</th>
<th>FA ($\gamma_1=1, \gamma_2=-1$)</th>
<th>AS ($\gamma_1=-1, \gamma_2=1$)</th>
<th>AA ($\gamma_1=-1, \gamma_2=-1$)</th>
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<td>SS ($\gamma_1=\gamma_2=1$)</td>
<td>FA ($\gamma_1=\gamma_2=-1$)</td>
<td>AS ($\gamma_1=-1, \gamma_2=1$)</td>
<td>AA ($\gamma_1=-1, \gamma_2=-1$)</td>
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<td>FFH, FFAS, FFAA, FFSS</td>
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<td>FFH, FFAS, FFAA, FFSS</td>
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<td>(b)</td>
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Fig. 3. Illustration of characterization of (a) Diagonal symmetry, (b) 90° rotational symmetry.
Fig. 3(b) shows $x[n]$ (which is $90^\circ$ rotationally symmetric) and $x[T_3n]$, $x[T_3^2n]$, and $x[T_5n]$ to illustrate that the above characterization holds. In the case of $90^\circ$ rotational symmetry, the center of symmetry can only be of type FF or HH. The center of symmetry in the signal in Fig. 3(b) is of type HH. Also, we can have an identity-symmetry or antisymmetry associated with the $T_5$ operation. This can be written as

$$x[n] = \gamma_{5^*}[T_5n + A_5c] = x[T_5^2n + 2c] = \gamma_{5^*}[T_5^2n + A_6c]. \quad (3a)$$

In z-domain

$$X(z) = \gamma_{5^*}(-A_6c)X[T_5] = z^{-2c}X[zT_5^2] = \gamma_{5^*}(-A_6c)X[zT_5^2]. \quad (3b)$$

$\gamma_5$ can be $+1$ or $-1$, thus giving two possibilities of symmetry. Combining this with the two types of the center of symmetry, the different types of $90^\circ$ rotational symmetries can be summarized as in Table I(c).

III. EFFECT OF FILTERING (CONVOLUTION) ON THE SIGNAL SYMMETRY

We generalize [2, Lemma 1] for the case of all four-fold signal symmetries as follows (as noted earlier, [2, Lemma 1] only considers the quadrantal symmetric case).

Proposition-1: When the impulse response of the filter and the input signal have the same symmetry type (quadrantal, diagonal, or $90^\circ$ rotational), then the filtered output signal also has the same symmetry type (quadrantal, diagonal, or $90^\circ$ rotational). The center of symmetry of the filtered output is the addition of the centers of symmetry of the input signal and the filter impulse response.

The above proposition can be verified easily from the z-domain expressions of the symmetries in Figs. (1b), (2b), and (3b). Also, in case of antisymmetries, the antisymmetry parameters as of the output signal are given by the products of the corresponding antisymmetry parameters of the input signal and the filter impulse response.

A. Design of Quincunx Filter-Banks With Filters Having Symmetries

Proposition-1 says that the filters in the Quincunx filter-banks should have the same symmetry type as the input. We just point out in this subsection that the method of transformations, as described in [8], gives us Quincunx filter-banks with quadrantal and diagonal symmetries, which also implies $90^\circ$ rotational symmetry.

Fact 1: If a signal has quadrantal and diagonal symmetries, then it also has $90^\circ$ rotational symmetry. This symmetry is also called octagonal symmetry.

This fact can be easily verified by noting that $T_1T_3 = T_5$ and $T_1T_4 = T_5^2$. The design method of [8] consists of designing a 1-D prototype product filter, and a 2-D transformation function $M(z)$, that satisfies $M(z) = M(\bar{z})$ (we refer the reader to [8] for more details). All the design examples in [8] use a 1-D zero-phase function for the product filter, and the 2-D transformation $M(z)$ is also chosen to be zero-phase. Now, we observe that, if we impose quadrantal or diagonal or $90^\circ$ rotational symmetry condition on $M(z)$, then the resulting filters in the 2-D filter-bank also have the same symmetry. We now consider the methods proposed in [8] to design $M(z)$, $m[\nu]n = \{m[n]n\}T$ is taken to be of the form $m[n, n] = m_1(n_1 + n_2)m_1(n_1 - n_2)$, where $m_1(n)$ is a 1-D function.

If we assume that $m_1(n_1) = m_1(-n_1)$, then $m[n_1, n_2]$ as designed above is octagonally symmetric. Actually, all the design examples in [8] satisfy the above conditions and thus give filters having octagonal symmetry, which are appropriate for preserving the signal symmetry (from Proposition-1). We note that this has not been explicitly observed in [8]. We also note that the center of symmetry of the filters is of type FF.

IV. EFFECT OF QUINCUNX DOWNSAMPLING ON THE SIGNAL SYMMETRY

We now analyze the behavior of the four-fold symmetries under Quincunx downsampling. We do this analysis by considering each symmetry separately.

A. Quadrantal Symmetric Input

Consider Fig. 1(a) with $x[n]$ being a quadrantly symmetric signal, as in (1) with center of symmetry $c = [c_1, c_2]^T$. The downsampled signal is given by $\nu_3[n] = x_1[Qu]$

$$\nu_3[n + 2Q^{-1}A_1c] = x_1[Q(n + 2Q^{-1}A_1c)] = x_1[Qu + 2A_1c].$$

Similarly, we can show that

$$\gamma_1(\nu_3[Q^{-1}T_1Qu + 2Q^{-1}A_1c]) = \nu_3[Qu + 2A_1c].$$

Now, we make the following important observations:

$$Q^{-1}T_1Q = T_3, \quad Q^{-1}T_1Q = T_4, \quad Q^{-1}T_1T_2Q = T_3T_4 = -I$$

$$A_3 = 2Q^{-1}A_1Q, \quad \text{and} \quad A_4 = 2Q^{-1}A_2Q. \quad (7)$$

Using this in (4)–(6), we have, with $d = Q^{-1}c$

$$\nu_3[n] = \gamma_1(\nu_3[T_3n + A_3d]) = \gamma_2(\nu_3[T_4n + A_4d])$$

Thus, $\nu_3[n]$ has diagonal symmetry with center of symmetry $d = Q^{-1}c = (1/2)[c_1 + c_2, c_1 - c_2]^T$. For diagonal symmetry, we require $d$ to be of type FF or HH. This puts a constraint on the center of symmetry $c$ of the input signal, which we now analyze. Since $c$ is the center of symmetry of a quadrantal symmetric signal, it can be of types FF, FH, HF, or HH. However, for $d$ to
be of type FF or HH, it can be verified that \( c \) should be of type FF. Thus, we can summarize this as follows.

**Proposition-2:** Consider a quadrantally symmetric input signal with center of symmetry \( c \). On downsampling this signal using the Quincunx sampling matrix \( Q \), the downsampled signal has diagonal symmetry with center of symmetry \( d = Q^{-1}c \), if \( c \) is of the type FF.

**B. Diagonally Symmetric Input**

Consider Fig. 1(a) with \( x[n] \) being a diagonally symmetric signal, as in (2), with center of symmetry \( c \).

Following a similar analysis as in Section III-A, and using (7), we have

\[
\begin{align*}
\gamma_3^4 n [Q^{-1}T_3Qn + A_1Qc] &= \epsilon_3^4 [n] \\
\gamma_4^4 n [Q^{-1}T_4Qn + A_2Qc] &= \epsilon_4^4 [n] \\
\gamma_3^3 \gamma_4^4 n [Q^{-1}T_3T_4Qn + 2Q^{-1}c] &= \epsilon_3^3 [n].
\end{align*}
\]

Using (7), and with \( d = Q^{-1}c \) (and also using the fact that \( Q = 2Q^{-1} \)), we can write (8)–(10) as

\[
\begin{align*}
\epsilon_3^4 [n] &= \gamma_3^4 n [T_1n + 2A_1d] = \gamma_4^4 n [T_2n + 2A_2d] \\
&= \gamma_3^3 \gamma_4^4 n [T_3T_4n + 2d].
\end{align*}
\]

\( \epsilon_3^4 [n] \) has quadrant symmetry with center of symmetry \( d = Q^{-1}c \). For quadrant symmetry, we require \( d \) to be of type FF, FH, or HH. Since \( c \) is the center of symmetry of a diagonally symmetric signal, it can be of types FF or HH. For both of these types of \( c \), it can be seen that \( d = Q^{-1}c \) is of type FF, FH, FF, or HH. Thus, we have the following.

**Proposition-3:** Consider a diagonally symmetric input signal with center of symmetry \( c \). On downsampling this signal by using the Quincunx sampling matrix \( Q \), the downsampled signal has quadrant symmetry with center of symmetry \( d = Q^{-1}c \) for any choice of \( c \) (the only possible choices for diagonally symmetric input are FF and FH).

**C. 90° Rotational Symmetric Input**

Consider Fig. 1(a) with \( x[n] \) being a 90° rotational symmetric signal, as in (3), with center of symmetry \( c \). Following a similar analysis as in Section III-A, and using the facts \( A_6 = JQ \) and \( A_6 = QJ \), where \( J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), we have

\[
\begin{align*}
\gamma_3^4 n [Q^{-1}T_5Qn + Q^{-1}JC] &= \epsilon_3^4 [n] \\
\gamma_4^4 n [Q^{-1}T_5Q + J] &= \epsilon_4^4 [n] \\
\gamma_3^3 \gamma_4^4 n [Q^{-1}T_5Qn + 2Q^{-1}c] &= \epsilon_3^3 [n].
\end{align*}
\]

Noting that \( Q^{-1}T_5Q = T_5, \ Q^{-1}T_5Q = T_5, \ Q^{-1}T_5Q = T_5 = -I \), we thus have from (11)–(13), with \( d = Q^{-1}c \)

\[
\epsilon_3^4 [n] = \gamma_3^4 n [T_3n + A_6d] = \epsilon_3^4 [T_3n + 2d] = \gamma_3^4 n [T_5n + A_6d].
\]

Thus, \( \epsilon_3^4 [n] \) has 90° rotational symmetry with the center of symmetry \( d = Q^{-1}c \). For 90° rotationally symmetry, we require \( d \) to be of type FF or HH. Since \( c \) is the center of symmetry of a 90° rotationally symmetric signal, it can be of types FF or HH. For \( d = Q^{-1}c \) to be of type FF or HH, \( c \) should be of type FF. Thus, we can summarize this as follows.

**Proposition-4:** Consider a 90° rotationally symmetric input signal with center of symmetry \( c \). On downsampling this signal by using the Quincunx sampling matrix \( Q \), the downsampled signal also has 90° rotationally symmetric input signal with center of symmetry \( d = Q^{-1}c \), if \( c \) is of type FF.

**V. CONCLUSION**

We provided a characterization of four-fold symmetries (quadrantial, diagonal, and 90° rotational) in 2-D signals. We showed that, after Quincunx downsampling, 1) a quadrantally symmetric input signal becomes a diagonally symmetric signal, 2) a diagonally symmetric input signal becomes a quadrantly symmetric signal, and 3) a 90° rotationally symmetric input signal remains 90° rotationally symmetric. We also showed that the method of transformations of [8] can be used to design Quincunx filter-banks with the filters having all of the four-fold symmetries.

**REFERENCES**


