On Filter Symmetries in a Class of Tree-Structured 2-D Nonseparable Filter-Banks

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Abstract—In one-dimensional (1-D) filter-banks (FBs), symmetries (or anti-symmetries) in the filter impulse-responses (which implies linear-phase filters) are required for symmetric signal extension schemes for finite-extent signals. In two-dimensional (2-D) separable FBs, essentially, 1-D processing is done independently along each dimension. When 2-D nonseparable FBs are considered, the 2-D filters (2-D signals in general) can have a much larger variety of symmetries (anti-symmetries) than the 1-D case. Some examples of 2-D symmetries possible are quadrantal, diagonal, centro, 4-fold rotational, etc. In this letter, we analyze the filter symmetries in a subclass of tree-structured 2-D nonseparable FBs, whose sampling matrices can be factored as a product of a Quincunx sampling matrix and a diagonal matrix. Within this subclass, we distinguish between two types and show that we can have diagonally symmetric filters in Type-I FBs and quadrantly symmetric filters in Type-II FBs. We then discuss how these FBs with quadrantly and diagonally symmetric filters can be used with a symmetric signal extension scheme on finite-extent signals.

Index Terms—Multidimensional filter banks, nonseparable filter banks.

I. INTRODUCTION

In the one-dimensional (1-D) case, filter-banks (FBs) with filters having symmetries or anti-symmetries,1 which lead to linear-phase FBs, are commonly used for image processing and coding applications, and design of such 1-D linear-phase FBs has been extensively addressed in the literature (see [1], [9], [10], and references therein). Such symmetric filters are also important in 1-D symmetric signal extension schemes—which are used to maintain critical sampling when the input signal is of finite extent [4]–[6]. In the two-dimensional (2-D) case, separable processing is commonly used in practice—wherein the filters and the sampling are separable. This essentially means that the 1-D FBs are used on each dimension independently. There is also a lot of interest in 2-D nonseparable FBs and their application to image coding and processing (see [1]–[3]). In the 2-D nonseparable case, linear-phase corresponds to centro-symmetry of the filter. Design of linear-phase (i.e., centro-symmetric) nonseparable FBs has been treated in [3] and [11]. However, in the 2-D case, much more symmetries are possible (other than centro-symmetry), like, e.g., quadrantal symmetry, diagonal symmetry, 4-fold rotational symmetry, etc. Fig. 1 shows examples of signals with the various types of symmetries mentioned above. Design of 2-D nonseparable FBs, with filters having symmetries as mentioned above, is an area not adequately addressed in the literature (to the best of our knowledge). One of the applications of FBs with filters having such symmetries is in symmetric signal extension schemes when used with nonseparable FBs, as we discuss later in this letter. We note that this variety of symmetries in the 2-D case is unlike the 1-D case, where only limited types of symmetries are possible.

Our main goal in this letter is to design 2-D nonseparable FBs with filters having quadrantal and diagonal symmetries. We restrict ourselves to a class of tree-structured FBs whose sampling matrices can be factored as a product of the corresponding matrices for 2-D signals. In this class, we distinguish between two types: Type-I, with sampling matrix factored as $\mathbf{M} = \mathbf{Q} \mathbf{\Lambda}$, and Type-II, with sampling matrix factored as $\mathbf{M} = \mathbf{\Lambda} \mathbf{Q}$. As an example, the hexagonal sampling matrix can be written in Type-I and Type-II forms as follows. Type-I: $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and Type-II: $\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

The rest of this letter is organized as follows: In Section II, we review the characterization of symmetries in 2-D signals (we note that the 2-D signal can also be the impulse response of a 2-D filter). In Section III, we show that Type-I tree-structured FBs can be designed with filters having diagonal symmetries, and Type-II FBs can have filters with quadrantally symmetric. Finally, in Section IV, we discuss symmetric signal extensions for finite-extent input 2-D signals, to illustrate an application of the tree-structured FBs with symmetric filters.

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1When we say a filter has a certain symmetry, we mean that the impulse-response of the filter has that symmetry.
and we can characterize a signal $x[n]$ that has diagonal symmetry with respect to center $c$ as

$$x[n] = x[T_3n + A_3c] = x[T_4n + A_4c] = x[T_3T_4n + 2c].$$

We note that we can distinguish between the cases where a) the center of symmetry falls midway between sample-points (i.e., $c \in (1/2)Z^2$), where $Z^2$ denotes the set of all two-component integer vectors with odd-valued integer components), and b) where the center of symmetry falls on a sample-point (i.e., $c \in Z^2$). This is analogous to the so-called “half-sample” and “full-sample” symmetries in the 1-D case [5].

Observing that $A_3 = 2Q^{-1}A_1Q$ and $A_4 = 2Q^{-1}A_2Q$, we have the following alternate characterization of diagonal symmetry:

$$x[n] = x[T_3n + 2Q^{-1}A_1QC] = x[T_4n + 2Q^{-1}A_2QC] = x[T_3T_4n + 2c].$$

The above characterizations can be written in $z$-domain as follows.

**Quadrantal symmetry:**

$$X[z] = z^{-2c}X(z^{-1}) = z^{-2A_1c}X(z^T_1) = z^{-2A_2c}X(z^T_2),$$

**Diagonal symmetry:**

$$X[z] = z^{-2c}X(z^{-1}) = z^{-2Q^{-1}A_1QC}X(z^T_1) = z^{-2Q^{-1}A_2QC}X(z^T_2).$$

All the above are identity-symmetries. We are also interested in anti-symmetries. This can be written as in the following for diagonal symmetry (For notation, we use the same subscript for $\gamma$ as the $T$-operation with which the $\gamma$ is associated):

$$X[z] = \gamma_3\gamma_4z^{-2c}X(z^{-1}) = \gamma_3z^{-2Q^{-1}A_1QC}X(z^T_1) = \gamma_4z^{-2Q^{-1}A_2QC}X(z^T_2).$$

$\gamma_3$ and $\gamma_4$ can each independently be $+1$ or $-1$, thus giving three types of diagonal anti-symmetries. Similar anti-symmetries can be formulated for the quadrantal case as well.

**III. Symmetries in Tree-Structured Filter-Banks**

In this section, we analyze the filter symmetries in the Type-I and Type-II tree-structured FBs (as mentioned in Section I) and show that we can have diagonally symmetric Type-I FBs and quadrantly symmetric Type-II FBs. First, we discuss the filter symmetries in a Quincunx FB, since this plays an important role in our discussions later.

**A. Filter Symmetries in Quincunx Filter-Banks**

We just point out that the method of transformations, as described in [8], gives us Quincunx FBs with quadrantal and diagonal symmetries. We note that this has not been explicitly observed in [8]. The design method of [8] consists of designing a 1-D prototype product filter, and a 2-D transformation function $M(z)$, that satisfies $M(z) = M(-z)$. All the design examples

**II. Characterization of 2-D Signal Symmetries**

We generalize the characterization of 2-D symmetries from [9], to include the case where the center of symmetry is not the origin. A 2-D signal is said to be symmetric if $x[Tn + b] = x[n]$ (identity-symmetry) or $x[Tn + b] = -x[n]$ (anti-symmetry). Here $b$ is a 2 x 1 vector, and $T$ is a non-singular matrix. The most commonly used $T$ matrices (and the ones we use in this letter) are: $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $T_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $T_4 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

(These can be interpreted as: $T_1 =$ Reflection about $x_1$ axis. $T_2 =$ Reflection about $x_2$ axis. $T_3 =$ Reflection about $x_1 = x_2$ diagonal. $T_4 =$ Reflection about $x_1 = -x_2$ diagonal). Note that $T_1T_2 = T_3T_4 = -I$.

We now only consider quadrantal and diagonal symmetries, as those are the only ones we deal with in this letter.

We can characterize a signal $x[n]$ that has quadrantal symmetry with respect to center $c \in (1/2)Z^2$ (where $Z^2$ denotes the set of all two-component integer vectors) as

$$x[n] = x[T_1n + 2A_1c] = x[T_2n + 2A_2c] = x[T_1T_2n + 2c].$$

**Notation:** Boldfaced lowercase letters are used to represent vectors, and boldfaced uppercase letters are used for matrices. $|A|$ denotes the absolute value of the determinant of the matrix $A$. Following the notation of [2], a vector $z = [z_0 \ z_1]^T$ raised to a matrix power $A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}$ (we only consider vectors of size 2 and matrices of size $2 \times 2$) is defined as follows: $z^A$ is a vector whose $i$th entry is $z_0{A_{00}}^i + z_1{A_{01}}^i$, where $i = 0, 1$. Throughout this letter, we will use $Q$ to denote the particular Quincunx sampling matrix $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Downsampling a 2-D signal using a sampling matrix $M$ is defined as $y[n] = x[Mn]$.

For convenience of notation, we use the following boldfaced letters to denote some constant matrices that arise frequently in this letter: $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, and $A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
in [8] use a 1-D zero-phase function for the product filter, and the 2-D transformation \( M(z) \) is also chosen to be zero-phase. Now, we observe that, if we also impose quadrantal or diagonal symmetry condition on \( M(z) \), then the resulting filters in the 2-D FB also have the same symmetry.

We now consider the methods proposed in [8] to design \( M(z) \), \( \mathbf{n} = [n_1 \ n_2]^T \) is taken to be of the form \( m(n_1 + n_2) = m_1(n_1 + n_2)n_1(n_1 - n_2) \), where \( m_1(n) \) is a 1-D function. If we assume that \( m_1(n) = m_1(-n) \), then \( m(n_1, n_2) \) as designed above is quadrantally as well as diagonally symmetric. Thus, the corresponding filters obtained using this transformation are quadrantally as well as diagonally symmetric.

### B. Filter Symmetries in Type-I Filter-Banks

Recall that in Type-I FBs, the sampling matrix \( \mathbf{M} = \mathbf{Q} \mathbf{A} \). We only consider the analysis filters (and similar discussion holds for synthesis filters). Let \( H_i(z) \) denote the analysis filters in a separable filter-bank, \( \mathbf{FB}_A \), with sampling matrix \( \mathbf{A} \). Let \( G_i(z) \) denote the analysis filters in a Quincunx filter-bank, \( \mathbf{FB}_Q \), with sampling matrix \( \mathbf{Q} \). Type-I FBs can be implemented as a cascade of \( \mathbf{FB}_A \) and \( \mathbf{FB}_Q \). Each analysis channel in Type-I filter-bank is shown in Fig. 3(a). This can be redrawn using noble identities, as shown in Fig. 3(b). The equivalent analysis filters in Type-I filter-bank are \( G_i(z)H_j(z^Q) \), where \( i = 0 \ldots |Q| - 1 \) and \( j = 0 \ldots |A| - 1 \). (Note that \(|Q| = 2 \) and \(|A| = N_0A_1 \).

As discussed in Section III-A, the Quincunx filters \( G_i(z) \) can be designed to have diagonal symmetry. \( \mathbf{FB}_A \) can be designed by designing two 1-D FBs. We now show that \( H_i(z^Q) \) has diagonal symmetry if its component 1-D filters are linear-phase. Let \( H_i(z) = P_0(z_0)P_i(z_1) \), where \( P_0(z_0) \) and \( P_i(z_1) \) are the analysis filters in a 1-D phase filter-bank, \( i = 0 \ldots A_0 - 1 \) and \( i_1 = 0 \ldots A_1 - 1 \). Thus, the 1-D filters can be written as \((\text{assuming real-coefficient filters}) \ P_0(z_0) = \gamma_0 z_0^{N_0}P_0(z_0^{-1}) \) and \( P_i(z_1) = \gamma_i z_1 P_i(z_1^{-1}) \), where \( N_0 \) and \( N_1 \) is the length of the 1-D filters, and \( \gamma_0, \gamma_i \) are \( \pm 1 \) (we refer the reader to [9] and [10] for detailed discussion on the design of 1-D linear-phase PR FBs). Thus, \( H_i(z) \) can be written as

\[
H_i(z) = \gamma_0 \gamma_i z^{-2k}H_i(z^{-1}) = \gamma_i z^{-2a^k}H_i(z^{T_1})
\]

Thus, \( H_i(z) \) is quadrantally symmetric with center of symmetry \( k \).

Now with \( z \rightarrow z^Q \), and on some manipulations using the facts \((z^A)^B = z^{AB}, 2Q^{-1} = Q, QT_2 = T_4Q, \) and \( QT_1 = T_3Q \), we have

\[
H_i(z^Q) = \gamma_0 \gamma_i z^{-2k}H_i(z^{-Q}) = \gamma_i z^{-2a^k}A_iQk' H_i(z^{T_1Q})
\]

or, denoting \( R_i(z) = H_i(z^Q) \), we have that \( R_i(z) \) has diagonal symmetry with center of symmetry \( k' \). Thus, \( G_i(z) \) and \( R_i(z) \) both have diagonal symmetry, it is evident from (4) that the overall filter \( G_i(z)R_i(z) \) also has diagonal symmetry. We summarize this as follows.

**Proposition-1:** The equivalent filters in a Type-I FB, \( G_i(z)H_j(z^Q) \), have diagonal symmetry if a) the Quincunx filters \( G_i(z) \) have diagonal symmetry, and b) the 1-D component filters of \( H_j(z) \) are linear-phase. Also, the symmetry/anti-symmetry of the Type-I filters depends on the symmetry/anti-symmetry of the corresponding 1-D filters.

### C. Filter Symmetries in Type-II Filter-Banks

In Type-II FBs, the sampling matrix \( \mathbf{M} = \mathbf{AQ} \). Using the same notation as in Section III-B, one channel of Type-II filter-bank can be implemented as Fig. 3(c). This can be redrawn, using noble identities, as shown in Fig. 3(d). The equivalent analysis filters in Type-II filter-bank are \( G_i(z^A)H_j(z^Q) \). Now, as we saw in Section III-B, \( H_j(z) \) has quadrantal symmetry if the component 1-D filters are linear-phase, and (from Section III-A), we can have \( G_i(z) \) to have quadrantal symmetry. Thus, with \( k \) as the center of symmetry of \( G_i(z) \), and with \( z \rightarrow z^A \), we have

\[
G_i(z^A) = z^{-2ak}G_i(z^{-1})
\]

and noting that \( \mathbf{A} \) and the \( T_j \)'s commute (since they are diagonal matrices), we have that \( S_j(z) = G_i(z^A) \) is quadrantally symmetric with center of symmetry \( k' = \mathbf{A}k \). Thus, we can summarize this as follows.

**Proposition-2:** The equivalent filters in a Type-II filter-bank, \( G_i(z^A)H_j(z^Q) \), have quadrant symmetry if a) the Quincunx filters \( G_i(z) \) have quadrant symmetry, and b) the 1-D component filters of \( H_j(z) \) are linear-phase.

### IV. APPLICATION TO SYMMETRIC INPUT SIGNAL EXTENSION METHODS

As an application of the tree-structured FBs discussed in this letter, we consider the symmetric signal extension method, which is used with finite extent signals. Consider a \([\mathbf{M}]\)-channel 2-D nonseparable FB, as shown in Fig. 2. Assuming that the finite extent 2-D input signal has a rectangular region of support (ROS), symmetric extension of the input gives us a quadrantally symmetric signal. The two main aspects involved in the analysis of symmetric signal extension is the effect of the filtering operation and the downsampling operation on...
the symmetry of the input signal. Regarding the effect of the filtering operation, the following can be easily verified from the z-domain expressions of the symmetries (3) and (4).

**Proposition-3:** When the impulse response of the filter and the input signal have the same symmetry type (quadrantal or diagonal), then the filtered output signal also has the same symmetry type (quadrantal or diagonal), with a possible change in the center of symmetry.

Also, in case of anti-symmetries (of the same type), the anti-symmetry parameters as in (5) of the output signal are given

\[ \gamma_0 = \gamma_0', \gamma_0'' \]

and

\[ \gamma_0 = \gamma_0', \gamma_0'' \]

where \( \gamma_0', \gamma_0'' \) are the anti-symmetry parameters of the filter impulse-response, and \( \gamma_0', \gamma_0'' \) are the anti-symmetry parameters of the input signal.

Now, first consider a Type-II filter-bank with filters having quadrantal symmetry. Thus, referring to Fig. 2, with a quadrantly symmetric input \( x[n] \), the filtered output \( x[n] \) also has quadrantal symmetry. We now analyze the effect of downsampling (for the case of Type-II filter-bank). Let the center of symmetry of the signal \( x[n] \) be \( c \). Consider the signal \( v_1[n] \) obtained by downsampling \( x[n] \) using sampling matrix \( M \)

\[ v_1[n] = x_2[Mn] \Rightarrow v_1[M^{-1}n] = x_1[n] \]

\[ \Rightarrow v_1[M^{-1}(n + 2A_1c)] = x_1[n + 2A_1c] \]

With \( n \rightarrow T_1n \), and due to symmetry in \( x_1[n] \), we have

\[ v_1[M^{-1}T_1n + 2M^{-1}A_1c] = x_1[T_1n + 2A_1c] = x_1[n] \]

Replacing \( n \rightarrow Mn \), we have

\[ v_1[M^{-1}T_1Mn + 2M^{-1}A_1c] = x_1[Mn] = v_1[n] \]

Now, since \( M = \Lambda Q \Rightarrow M^{-1}A_1M = Q^{-1}A_1Q \). Also, \( M^{-1}T_1M = Q^{-1}T_1Q = T_3 \). So, with Type-II sampling matrix \( M \),

\[ v_1[T_3n + 2Q^{-1}A_1Qd] = v_1[n] \]

where \( d = M^{-1}c \). Similarly observing that \( M^{-1}T_2M = Q^{-1}T_2Q = T_4 \), and \( M^{-1}T_1T_2M = T_3T_4 = I \), we have

\[ v_1[n] = v_1[T_3n + 2Q^{-1}A_1Qd] \]

\[ = v_1[T_3n + 2Q^{-1}A_2Qd] \]

\[ = v_1[T_3T_4n + 2d] \]

i.e., \( v_1[n] \) has diagonal symmetry with the center of symmetry \( d = M^{-1}c \). Thus, we have the following.

**Proposition-4:** For a Type-II filter-bank with filters having quadrantal symmetry, when the input is quadrantly symmetric, then the (filtered and downsamplled) subband signals are diagonally symmetric.

Now considering a Type-I filter-bank with filters having diagonal symmetries, and following a very similar analysis as above, we have the following.

**Proposition-5:** For a Type-I filter-bank with filters having diagonal symmetry, when the input is diagonally symmetric, then the (filtered and downsamplled) subband signals are quadrantly symmetric.

So, based on Proposition-4 and 5, we can use cascades of Type-II and Type-I filter-bank structures (with Type-II for odd Stages 1, 3, …, and Type-I for even stages 2, 4, …) with a symmetric input extension, to maintain the symmetries in a subband image coding schemes using nonseparable FBs.

**V. Conclusion**

In this letter, we studied the filter symmetries in a class of tree-structured 2-D nonseparable FBs whose sampling matrices can be factored as a product of the Quincunx sampling matrix \( Q \) and a diagonal matrix. Within this class, we distinguished between two types and showed that we can have diagonally symmetric Type-I FBs and quadrantly symmetric Type-II FBs. We then analyzed the symmetric signal extension scheme and showed that filters having quadrantal and diagonal symmetries are required to maintain the symmetries in the subband signals. More general design methods for nonseparable FBs with filters having various symmetries is an area for further research.

**References**


