The present work investigates the influence of property variations of air in laminar forced convection with entrance effect. Two-dimensional micro-sized geometry (with axisymmetry) with constant wall heat flux boundary condition is considered to predict flow behaviour and thermal development. The continuum-based conservation equations are numerically solved to account for non-rarefaction scaling effects due to variations in fluid properties. At the microscale, results for Nusselt number show significant deviation from conventional theory that does not consider additional mechanisms that surface. The effect of property variation in Graetz problem for low subsonic flow is also studied.

1. Introduction

High heat flux dissipation technique [1] using liquid/gas as coolant is suitable for compact microscale devices. Over last few years, industries are focusing on miniaturized, high-performance and high-speed electronic devices like MEMS, MST, µ-TAS. These devices are having advanced applications in diverse areas viz. micro-electronic, aviation, metrology, micro-robot, micro-satellite, micro-engine and bioengineering. Micro-convection has been identified as research area in transport phenomena, therefore, understanding its characteristics is important. In past several researchers [2–5] have experimentally and theoretically investigated heat transfer characteristics at microscale. Their experimental investigation indicates that conventional correlations are no longer applicable. The results are deviated due to geometric configuration, fluid properties, surface roughness and viscous dissipation. To explain these departures from known characteristics, modification to the boundary conditions is given by scaling effect [6]. The most updated state-of-the-art review of single phase micro-convection has been illustrated by Morini [7]. Steinke and Kandlikar [8] have extensively reviewed literature on experimental investigation of microscale fluid flow and found diversity in experimental results from conventional theory. Sobhan and Garimella [9] prepared a comparative analysis to focus on anomalies and deviations from the behaviour expected for conventional channels. In comprehensive review, Mahulikar et al. [10] reported that the deviations from expected behaviour were attributed to scaling effects that surface out at microscale.

The scaling effect for gas micro-convection is classified into rarefaction and non-rarefaction. The rarefaction scaling effect arises because of failure of continuum assumption. This breakdown is characterized by Kn, defined as ratio of λ and Dh, which is used to explain deviation of fluid from continuum behaviour (rarefaction effect). For Kn < 0.001, fluid is assumed to be continuum, which is modelled by Navier–Stokes equation with no slip boundary condition. For Kn > 10, fluid is assumed a free molecular flow, which can be modelled with collisionless Boltzmann equation. For 0.001 < Kn < 0.1, flow regime is slip and first order velocity slip/temperature jump boundary condition is applied to the Navier–Stokes equation. Microdevices work in the range of 0.001 < Kn < 0.1 and flow is considered with slip boundary condition which lies nearer to continuum. The similarity between rarefied and micro-flow has prompted several authors to investigate along this line, however, this is not a complete similarity [11].

Non-rarefaction effects do not require a modification of Navier–Stokes equation. Therefore, it includes those terms which are prominent at microscale and often neglected due to lack of importance. These terms are axial conduction, viscous dissipation, channel surface condition, compressibility effect and property variation. In micro-convection, temperature gradients along and across the flow are steeper due to small m and low Re and therefore, enable high q* because of high surface area. The effects of variation in fluid properties are much stronger, hence, their relative role in micro-convection is significant.

1.1. Review of past work

The fluid flow and heat transfer characteristics in micro-devices with property variation reveal significant differences in
quality of convection. It has been observed that property ratio method [12] for correcting Nu fails to predict convection for large \( q_{w}^{*} \). Further asymptotic theory [13,14] is found to be better applicable for small \( q_{w}^{*} \) and large Re and not suitable for microflows. The role of property variation in micro-convection within continuum regime is investigated during last few years. Mahulikar et al. [15] numerically illustrated the role of gas density variation in \( \text{<unsigned math expression>} \) and mass flow rate is obtained as: \( \left( \frac{\partial}{\partial z} \right) q_{w}^{*} = 0 \). Hence, radial convection can not be neglected for determining convection characteristics [19].

Experimentally determined Nu for micro-flows was found to be less [2,5] whereas, some researchers [3,4] found its value greater than classical theory. In conventional duct Nu remains constant for laminar forced convection with uniform peripheral wall heat flux [25], however, for gas micro-flow Nu was found to vary significantly with respect to Re [16]. This has been attributed to variation of thermophysical fluid properties, due to large temperature gradients [15,19]. From the literature survey, it is observed that forced laminar gas micro-convection due to \( \rho(p,T), C_{p}(T), \mu(T) \) and \( k(T) \) variations with entrance effect is unexplored. The investigations by Mahulikar and Herwig [16–19] in the recent past have been undertaken to understand the physical mechanism of micro-convection. The results presented reveal the significance of property variation and their sensitivities in micro-convection. Further the model extends applicability of computationally inexpensive continuum model to higher Kn situations.

2. Physical model and boundary conditions

Fig. 1 shows schematic of 2D circular tube subjected to different wall heat flux boundary conditions.

Case 1. Entrance length problem: with \( q_{w}^{*} = 7.5 \text{ W/cm}^2 \) across \( L_1 \) and \( L_2 \).

Case 2. Graetz problem: with \( q_{w}^{*} = 0 \) and 7.5 W/cm² across \( L_1 \) and \( L_2 \) respectively.

The following four thermal and flow boundary conditions are incorporated.

1. **Inlet**: Flat velocity profile: \( u(r,0) = u_{in}(r) = u_{in,m} = 20 \text{ m/s} \) and flat temperature profile: \( T(r,0) = T(r) = T_{in,m} = 5{^\circ}\text{C} \). The subscript \( '0' \) refers to the value of parameter at the axis of micro-tube and mass flow rate is obtained as: \( m = \rho \pi D^2 u_{in,m} \).

2. **Outlet**: \( p_{exit} = 1.013 \times 10^5 \text{ Pa} \) (atmospheric pressure) and \( \nu_{exit} = 0 \) since, \( \partial u/\partial z = 0 \).

3. **Wall**: The channel walls are non-porous rigid with \( k \) and \( \rho(T) \) variations. \( \nu_{wall} = 0 \).

4. **Axis**: Symmetric boundary conditions are applied at the axis of micro-tube, hence, \( \partial u/\partial r = \partial T/\partial r = \partial \rho/\partial r = 0 \) and \( \nu = 0 \).

Inlet boundary condition reveals the role of property variation mixing with entrance effects. From \( z = 0 \) and onwards downstream (i.e. throughout entrance region \( = L_1 + L_2 \) \( \rho(p,T), C_{p}(T), \mu(T) \) and \( k(T) \) variations for non-reacting air is modelled.

2.1. Mathematical formulation

The 2D cylindrical co-ordinate (with axisymmetry) continuum-based governing differential equations together with ideal gas equation are numerically solved considering \( p, C_{p}, \mu \) and \( k \) variations. Continuity:

\[
\nabla \cdot (\rho \nabla \Phi) + \rho \cdot (\partial \Phi / \partial r) + (\rho \cdot v) + \rho \cdot (\partial \nu / \partial z) + u \cdot (\partial \rho / \partial z) = 0
\]

(1)
The $u$ and $v$ are the velocities in $z$ (axial coordinate) and $r$ (radial coordinate) direction. The term containing $\nabla \cdot v = 1/r \cdot \partial v/\partial r + \partial u/\partial z$ is incorporated in momentum equation for density variation [27]. Assumptions prescribed in the numerical computation are: flow is laminar, incompressible, steady and non-isothermal. The fluid is considered to be single phase, Newtonian and fluid modeling is within continuum region. Viscous dissipation ($\mu \cdot \Phi_v$) and compressibility terms in energy equation are neglected. Continuity, Navier–Stokes, energy and ideal gas equations are solved with proper relationship for fluid properties which shows interesting behaviour of energy transfer in a moving fluid.

### 3. Results and discussions

The convective heat transfer in micro-tube is characterized in terms of $Nu$ and interpreted as dimensionless temperature gradient at the surface. It shows the relative significance of heat transfer by the two mechanisms, i.e. axial convection and radial conduction Therefore, $Nu$ for a constant wall heat flux boundary condition reduces to,

$$Nu = \frac{q_w}{D (T_w - T_m)} k_m$$

The following data is fixed for uniformity: $R = 375 \mu m, L = 75 \text{ mm}$ and $T_{in} = 278 K$. For some cases results are reproduced with combination of $\rho(p,T), C_p(T), \mu(T)$ and $k(T)$ and their variations for different $Re_m, L/D$ and $q_w$. For air, $\mu(T)$ variation is given as

$$\mu(T) = 1.462 \times 10^{-6} T^{1.5} (T + 112) \text{ kg/m s}, \text{ where, } T \text{ is in K}$$

and $k(T)$ variation is given as [28],

$$k(T) = 2 \times 10^{-3} T^{1.5} (T + 112) \text{ (where, } T \text{ is in K})$$

The $C_p(T)$ is a function of temperature and modelled using sixth-order single piece linear polynomial and scaled between 0 and 1. As shown in Fig. 2 smooth curves are fitted over the range of temperature 200–2000 K within accuracy of 0.12%, which is justified [26].

### 3.1 Physical effects of variation in gas properties

The $\rho(p,T)$ and $\mu(T)$ variations have indirect effect through velocity $(u,v)$ profiles on $Nu$ whereas, $C_p(T)$ and $k(T)$ variations have direct effect through fluid temperature field.
3.1.1. Role of $\rho(p,T)$ variation

Fig. 3(a) illustrates variation in $Nu$ along simultaneously developing flow (for $\phi = 750 \mu m$ tube) with "$q_{\text{in}}^o$ = const." boundary condition considering properties variations. The dashed curve shows constant properties solution which is independent of $q_{\text{in}}^o$ and two other curve at the bottom shows $\rho(p,T)$ variation with different $q_{\text{in}}^o$.

For $\rho(p,T)$ variation, the radial velocity is zero at $r = 0$ and at $r = R$, and is radially-inward in between axis and wall. The mean velocity is $\nu_m = -39 \text{ cm/s}$ and $-2.5 \text{ cm/s}$ at $z/D = 0.5$ and 15, respectively. The $-$ve sign indicates inward radial velocity and its magnitude increases with radius. The cause of inducing radial flow is mean radial velocity and is given by,

$$\nu_m = \frac{2}{R^2} \int_0^R \nu(r) \cdot r \cdot dr$$

For the case of heated air, $Nu$ performance with $\rho$ variation can be determined jointly as:

(i) Radially-inward flow is acting in same direction of inward diffusion of temperature due to $q_{\text{in}}^o$. Therefore, radially-inward convection due to $\nu_m$ actually increases $Nu$.

(ii) But this radially-inward flow causes axial velocity profile to be sharpened (termed as hydrodynamic development of flow) [refer Fig. 4]. The sharpening of $u(r)$ profile results in lowering the axial velocity of moving particles close to wall. For $\nu(r) < 0$, radially-inward flow promotes convection but sharpened $u(r)$ profile degrades convection. Fig. 4 also shows that sharpening of $u(r)$ decreases $(\nu l/\nu_w)$ along the flow, thus, $u(r)$ profile is less effective in axial transport of the imposed $q_{\text{in}}^o$, thereby degrading convection.

(iii) The $\rho$ near wall (0.38 kg/m$^3$) is lower than at axis of tube (1.22 kg/m$^3$, both values are at $z/D = 15$) which degrades convection. Gas acceleration due to reduce in $\rho$ leads to change velocity profile in magnitude as well as in shape [refer Fig. 4].

Therefore, in the initial part of boundary layer formation net $Nu$ reduces abruptly due to sharpening of $u(r)$ profile. It is also observed that $Nu$ is minimum at $z/D = 50$ due to following reasons: (a) above-mentioned contradictory mechanism (i) and (ii), (b) induced outward $\nu_m$, (c) maximum value of $(T_w - T_m)$ [refer Fig. 5]. The minimum $Nu (=4.0425)$ differs from constant properties solution $(Nu_{CP})$ by 7.4% showing the sensitivity of $\rho(p,T)$ variation.

The result shown in Fig. 3(a) indicates slight increase in $Nu$ towards the end of channel. The reason behind this is outward $\nu_m$ with an induced outward velocity of 0.8 cm/s. Fig. 3(a) also shows comparison of $\rho(p,T)$ variation due to $q_{\text{in}}^o = 7.5$ and 10 W/cm$^2$ and it predicts that $Nu_{\text{max}}$ is still lower in 10 W/cm$^2$ case. This has been attributed to: (i) lower density and axial mass flux near the wall and (ii) increase in the value of $(T_w - T_m)$ from 573 to 773 K due to high $q_{\text{in}}^o$.

3.1.2. Role of $\mu(T)$ variation

Fig. 3(a) illustrates variation in $Nu$ along flow due to $q_{\text{in}}^o = 7.5 \text{ W/cm}^2$ considering $\mu(T)$ variation. For the case of heated air when entrance effects are considered, the role of $\mu(T)$ variation is to decrease $Nu$ compared to the constant properties solution. There are radially-inward velocities which promote $Nu$, but due to sharpening of $u(r,z)$ and $l(r,z)$ profile (resulting from radially-inward flow) $Nu$ decreases gradually along the flow.

For the case of heated air, the effect of hydrodynamic development on $u(r,z)$ and $l(r,z)$ profiles is same as that of $\mu(T)$ variation of air. For $q_{\text{in}}^o > 0$, the effect of $\mu(T)$ variation is to sharpen $u(r)$ profile by inducing radially-inward $\nu(r)$. Due to hydrodynamic development of $u(r)$ profile, induced $\nu_m$ reduces from $-6.8 \text{ cm/s}$ at $z/D = 5$ to $-1.2 \text{ cm/s}$ at $z/D = 20$, therefore radially-inward flow is reduced. This provides less sharpening effect of $u(r)$ profile, hence, $Nu_{\text{max}}$ is higher for $\mu(T)$ compared to $\rho(p,T)$ variation. The minimum $Nu = 4.19$ differs from constant properties solution $Nu = 4.81$ by 4% showing the sensitivity of $\mu(T)$ variation.
Fig. 3 reveals that combine effect of $q(p,T)$ and $l(T)$ variations is to reduce $Nu$ and its rate of decrease aggravates which can be interpreted as follows:

(i) Hydrodynamic-flow development is further enhanced due to induced radially-inward flow with $q(p,T)$ and $\mu(T)$ variations.

Fig. 3(b) reveals that combine effect of $\rho(p,T)$ and $\mu(T)$ variations is to reduce $Nu$ and its rate of decrease aggravates which can be interpreted as follows:
Fig. 4. Variation of axial velocity profile along the flow due to $\rho(p,T)$, $\rho(p,T)$ and $\mu(T)$ variations at several locations for $u_{in} = 20$ m/s and $q_{in}^* = 7.5$ W/cm².

Fig. 5. The $(T_w - T_m)$ versus $z/D$ due to combination of $\rho(p,T)$, $C_p(T)$, $\mu(T)$ and $k(T)$ variations for $u_{in} = 20$ m/s and $q_{in}^* = 7.5$ W/cm².
(ii) The \( \rho(p, T) \) variation renders sharpening of \( u(r, z) \) and gas-viscosity increases with temperature that thickens growth of boundary layer. The effect of viscosity penetration results in retardation of fluid particles which causes more resistance to flow and increases boundary layer thickness and eventually decreases \( h \).

(iii) Sharpening of \( u(r) \) profile due to \( \rho(p, T) \) and \( \mu(T) \) variations decreases \( \frac{\Delta u}{\Delta r} \) compared to \( \rho(p, T) \), thereby reducing \( Nu \) [refer Fig. 4].

It has been observed that, rate of decrease of \( Nu \) is aggravated due to \( \rho(p, T) \) and \( \mu(T) \) variations. This depends on strength of induced inward radial convection and its dominance is shown in Fig. 3(b). The minimum \( Nu \) (=3.65) differs from constant properties solution by 16.3% showing the sensitivity of combine \( \rho(p, T) \) and \( \mu(T) \) variations. Hydrodynamic flow development due to \( \mu(T) \) variation (distorted \( u(r, z) \) profile) also exits in conventional channel. But high Re and low \( q_w^* \) reduces \( u(r, z) \) distortion along the flow however, low Re and high \( q_w^* \) in micro-flow produces \( u(r, z) \) distortion by inducing \( r(z) \) variation. Thus, flow stability is affected by induced radial flow due to property variations [14]. Table 1 shows various flow properties and dimensionless numbers at different locations of geometry \((L/D = 100)\). The use of micro-tube diameter and length ensure that highest \( Kn \) in the computational domain is lower by one order of magnitude from, 0.001. The value of \( Kn \) is highest at the exit of tube, where \( \rho \) is lowest, and exit value of \( Kn \) increases with \( q_w^* \). The highest \( M \) at the exit is less than 0.3; hence, flow can be treated as incompressible. The \( Re \) decreases in downstream direction, since, air viscosity increases with temperature. The maximum \( T_{w, exit} \) is less than air dissociation temperature and \( \rho \) values at exit can differ from 1.225, by about 70%.

3.1.3. Role of \( Cp(T) \) variation

Fig. 3(a) illustrates variation in \( Nu \) along simultaneously developing flow for \( q_w^* = \text{const.} \) due to \( Cp(T) \) variation. For the case of air being heated, the role of \( Cp(T) \) variation is to decrease \( Nu \) due to sharpening of \( u(r, z) \), \( T(r, z) \) profile and it reduces \( T_w \) along the flow. It is observed that the effect of \( Cp(T) \) variation also results in reduction of \( T_m \). This effect actually decreases \( Nu \) as compared to constant properties solution [refer Eq. (5)]. But high \( Cp(T) \) near the wall reduces \( T_w \) for \( q_w^* > 0 \) and this reduction in \( T_w \) compensates for the decrease in \( T_m \). This phenomenon decreases \( (T_w - T_m) \), hence, \( Nu \) increases along the heated flow [refer Fig. 5 and Eq. (5)]. The \( T_m \) is a representative of total energy of flow at a particular location in micro-tube (enthalpy-average temperature of bulk) and is given by,

\[
T_m = \int_0^L \rho \cdot u \cdot C_p \cdot T \cdot r \cdot dr / \int_0^L \rho \cdot u \cdot C_p \cdot r \cdot dr
\]

From numerical simulation, it is observed that physical effect of \( Cp(T) \) variation is more through temperature field than velocity field. Axial temperature gradient \((\partial T/\partial z) \) changes with \( Cp(T) \) and it increases the variation in property along the flow. For \( Cp(T) \) variation rate of decrease of \( Nu \) compared to constant properties solution is alleviated, since, it reduces \( T_w \) and \( (T_w - T_m) \) [refer Fig. 5]. Therefore, \( Nu_{exit} \) obtained with \( Cp(T) \) variation is higher than constant properties solution, by about 4%. Fig. 3(a) shows increase in \( Nu_{exit} \) with increasing \( q_w^* \) due to \( Cp(T) \) variation alone for given \( Re_w \). The reason behind this is \( q_w^* \) causes more reduction in axial temperature gradient, \( T_m \) and \( (T_w - T_m) \).

Fig. 3(b) illustrates combine \( \rho(p, T) \) and \( Cp(T) \) variations due to \( q_w^* = 7.5 \text{ W/cm}^2 \). The rate of decrease of \( Nu \) is higher than constant properties solution, since role of \( \rho(p, T) \) is dominant in inducing inward radial flow and sharpening of \( u(r, z) \) and \( T(r, z) \) profile. But \( Cp(T) \) variation is prominent at high temperature region, i.e. towards the end of channel. The \( \rho(p, T) \) and \( Cp(T) \) variations have decreasing effect on (i) sharpening of temperature profile [refer Fig. 6] (ii) \( \partial T/\partial z \) (iii) \( Nu \) and (iv) \( (T_w - T_m) \). The effect of \( Cp(T) \) is to reduce radial inward convection, hence, \( \rho(p, T) \) and \( Cp(T) \) variations contribute to increase in \( Nu \) compared to \( \rho(T) \) variation alone. The minimum \( Nu \) differs from constant properties solution by 4.8% showing the sensitivity of \( \rho(p, T) \) and \( Cp(T) \) variations. The different values of heat flux due to \( \rho(p, T) \) and \( Cp(T) \) variations viz. \( q_w^* = 5, 7.5 \text{ W/cm}^2 \) are calculated and tabulated in Table 1.

### Table 1

<table>
<thead>
<tr>
<th>Combination of properties variation with different ( q_w^* )</th>
<th>Flow properties at various location of geometry</th>
<th>Outlet</th>
<th>( Re_{exit} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( \rho(p, T), Cp(T), \mu(T) ) and ( k(T) ):</td>
<td>Inlet ( T_w, T_m )</td>
<td>( \rho_m, C_{p,m} )</td>
<td>( \mu_m, 10^3, k_m, 10^3 )</td>
</tr>
<tr>
<td>(i) ( u_m = 20 \text{ m/s}, q_w^* = 5 \text{ W/cm}^2 )</td>
<td>366, 278</td>
<td>845, 671</td>
<td>1165, 1042</td>
</tr>
<tr>
<td>(ii) ( u_m = 20 \text{ m/s}, q_w^* = 7.5 \text{ W/cm}^2 )</td>
<td>1.19, 1004</td>
<td>0.53, 1068</td>
<td>0.34, 1148</td>
</tr>
<tr>
<td>(b) ( \rho(p, T), Cp(T), \mu(T) ):</td>
<td>Inlet ( T_w, T_m )</td>
<td>( \rho_m, C_{p,m} )</td>
<td>( \mu_m, 10^3, k_m, 10^3 )</td>
</tr>
<tr>
<td>(i) ( u_m = 20 \text{ m/s}, q_w^* = 5 \text{ W/cm}^2 )</td>
<td>183.5, 26</td>
<td>323.4, 50.5</td>
<td>427.6, 70</td>
</tr>
<tr>
<td>(ii) ( u_m = 20 \text{ m/s}, q_w^* = 7.5 \text{ W/cm}^2 )</td>
<td>407, 278</td>
<td>1075, 857</td>
<td>1534, 1388</td>
</tr>
<tr>
<td>(c) ( \rho(p, T) ) and ( Cp(T) ):</td>
<td>Inlet ( T_w, T_m )</td>
<td>( \rho_m, C_{p,m} )</td>
<td>( \mu_m, 10^3, k_m, 10^3 )</td>
</tr>
<tr>
<td>(i) ( u_m = 20 \text{ m/s}, q_w^* = 5 \text{ W/cm}^2 )</td>
<td>1.19, 1004</td>
<td>0.56, 1068</td>
<td>0.35, 1146</td>
</tr>
<tr>
<td>(ii) ( u_m = 20 \text{ m/s}, q_w^* = 7.5 \text{ W/cm}^2 )</td>
<td>375, 278</td>
<td>320.9, 24.2</td>
<td>426.3, 24.2</td>
</tr>
<tr>
<td>(d) ( \rho(p, T) ):</td>
<td>Inlet ( T_w, T_m )</td>
<td>( \rho_m, C_{p,m} )</td>
<td>( \mu_m, 10^3, k_m, 10^3 )</td>
</tr>
<tr>
<td>(i) ( u_m = 20 \text{ m/s}, q_w^* = 5 \text{ W/cm}^2 )</td>
<td>1.19, 1004</td>
<td>0.55, 1068</td>
<td>0.35, 1146</td>
</tr>
<tr>
<td>(ii) ( u_m = 20 \text{ m/s}, q_w^* = 7.5 \text{ W/cm}^2 )</td>
<td>178.9, 24.2</td>
<td>178.9, 24.2</td>
<td>178.9, 24.2</td>
</tr>
</tbody>
</table>
3.1.4. Role of $k(T)$ variation

The role of $k(T)$ [13,20] and additional physical mechanism of $k(T)$ variation is elucidated in the literature [16,18]. Fig. 3(a) illustrates variation in $Nu$ along simultaneously developing flow for “$q''_w = \text{const.}$” due to $k(T)$ variation. For the case of heated air, the role of $k(T)$ variation is to decrease $Nu$ due to radially-inward $r(z)$ and sharpening of $u(r,z)$ and $T(r,z)$ profile. The rate of decrease of $Nu$ is less than constant properties solution which is explained as follows:

The $k(T)$ induces outward radial velocity which counteracts with inward diffusion from wall. Hence, $Nu$ decrease rate is alleviated due to diminishing radially-inward flow.

Heat flow at cross section is given by,

$$q''_w = k_w(\partial T/\partial r)_w.$$  \hspace{1cm} (10)

For constant $q''_w$, increasing $k_w$ along heated flow reduces corresponding temperature gradient near the wall. Higher $k$-fluid near the wall is more effective in convecting away the imposed thermal boundary condition at the wall, hence, promotes $Nu$ [refer Eq. (10)].

Fig. 6 shows that $k$ variation flattens the temperature profile thus; 'U'-shaped $k(r)$ profile promotes convection relative to $Nu_{\text{cr}}$. The axial gradient is given by [16,18],

$$|\partial T_m/\partial z| = 4(q''_w/D)/[(\rho \cdot m \cdot u_m \cdot C_{pm})(\partial k_m/\partial z)].$$  \hspace{1cm} (11)

The $k(z)$ variation induces axial conduction which significantly affect on micro-convection by increasing $\partial k_m/\partial z$. The higher $\partial k_m/\partial z$ along the heated flow provides large $\partial T_m/\partial z$ which tends to increase $Nu$ [refer Eq. (11)]. Therefore, $k$ variation along and across the flow has the same effect on micro-convection and promotes $Nu$ after minima compared to constant properties solution.

Fig. 3(b) illustrates combine $\rho(p,T)$ and $k(T)$ variations due to $q''_w = 7.5 \text{ W/cm}^2$. After initial decrease of $Nu$, role of $k(T)$ is dominant in reducing $T_w$, than inducing inward-radial flow by $\rho(p,T)$. The combine effect of $\rho(p,T)$ and $k(T)$ variations after minima is to induce radially-outward velocities [16]. The magnitude of $v_m$ is 1.5 cm/s at $2D = 30$ and this causes outward flow which flattens $u(r)$ profile. Thus, it promotes convection due to faster moving particle close to the boundary. The $\rho(p,T)$ and $k(T)$ variations also reduces $(T_w-T_m)$ which differs from increase in $(T_w-T_m)$ for constant properties solution, thereby increases $Nu$ [refer Fig. 5 and Eq. (5)]. The $k(T)$ variation and combine $\rho(p,T)$ and $k(T)$ variations decreases the sharpening of temperature profile. This flattens the temperature profile which promotes the convection (refer Fig. 6).

Therefore, $Nu$ is seen to go through minima and increase in $Nu$ after minima is a result of dominance of $k(T)$ variation which is clearly seen in Fig. 3(b). The $Nu = 4.55$ at exit differs from $Nu_{\text{cr}}$ by 4.1% which shows sensitivity due to $\rho(p,T)$ and $k(T)$ variations.

It is observed that $Nu$ is seen to go through minima in case of (i) $\rho(p,T)$, (ii) $\rho(p,T)$ and $k(T)$, (iii) $\rho(p,T)$ and $C_p$, (iv) $\rho(p,T)$ and $k(T)$ and (v) $\rho(p,T)$, $\rho(T)$, $C_p(T)$ and $k(T)$. The following trend of $Nu$ minima is summarized: $Nu_{\text{min,(i)}} > Nu_{\text{min,(ii)}} > Nu_{\text{min,(iii)}} > Nu_{\text{min,(iv)}} > Nu_{\text{min,(v)}} > Nu_{\text{min,(v)}}$.

For $\rho(p,T)$, $C_p(T)$, $\rho(T)$ and $k(T)$ variations, net $Nu$ increases after minima. This is due to contradictory mechanism as follows: (i) the induced inward radial convection due to $\rho(T)$ variation, (ii) reduce in $T_w$ and $(T_w-T_m)$ due to $C_p(T)$ variation and (iii) role of $k(T)$ variation is to increase induced radially-outward flow due to $\rho(p,T)$ variation.

The variation in $Nu$ due to $\rho(p,T)$, $\rho(p,T)$ and $C_p(T)$ variations and all properties variations shows an upward trend and seem to approach constant properties solution. It is found that for increase in $L/D$ from 100 to 125, an upward trend remains same and change in $Nu_{\text{exit}}$ is within 0.9% for above three cases. Table 2 shows $Nu_{\text{exit}}$ and 10 W/cm², provides approx. same value of $Nu = 4.22$ at exit, which is less than $Nu_{\text{cr}}$. 

**Fig. 6.** Variation of radial bulk mean temperature at $z/D = 50$ due to combination of $\rho(p,T)$, $C_p(T)$, $\rho(T)$ and $k(T)$ variations for $u_m = 20 \text{ m/s}$ and $q''_w = 7.5 \text{ W/cm}^2$. 

and 10 W/cm², provides approx. same value of $Nu = 4.22$ at exit, which is less than $Nu_{\text{cr}}$. 

**Fig. 3(b).** Illustrates combine $\rho(p,T)$ and $k(T)$ variations due to $q''_w = 7.5 \text{ W/cm}^2$. After initial decrease of $Nu$, role of $k(T)$ is dominant in reducing $T_w$, than inducing inward-radial flow by $\rho(p,T)$. The combine effect of $\rho(p,T)$ and $k(T)$ variations after minima is to induce radially-outward velocities [16]. The magnitude of $v_m$ is 1.5 cm/s at $2D = 30$ and this causes outward flow which flattens $u(r)$ profile. Thus, it promotes convection due to faster moving particle close to the boundary. The $\rho(p,T)$ and $k(T)$ variations also reduces $(T_w-T_m)$ which differs from increase in $(T_w-T_m)$ for constant properties solution, thereby increases $Nu$ [refer Fig. 5 and Eq. (5)]. The $k(T)$ variation and combine $\rho(p,T)$ and $k(T)$ variations decreases the sharpening of temperature profile. This flattens the temperature profile which promotes the convection (refer Fig. 6).

Therefore, $Nu$ is seen to go through minima and increase in $Nu$ after minima is a result of dominance of $k(T)$ variation which is clearly seen in Fig. 3(b). The $Nu = 4.55$ at exit differs from $Nu_{\text{cr}}$ by 4.1% which shows sensitivity due to $\rho(p,T)$ and $k(T)$ variations.

It is observed that $Nu$ is seen to go through minima in case of (i) $\rho(p,T)$, (ii) $\rho(p,T)$ and $k(T)$, (iii) $\rho(p,T)$ and $C_p$, (iv) $\rho(p,T)$ and $k(T)$ and (v) $\rho(p,T)$, $\rho(T)$, $C_p(T)$ and $k(T)$. The following trend of $Nu$ minima is summarized: $Nu_{\text{min,(i)}} > Nu_{\text{min,(ii)}} > Nu_{\text{min,(iii)}} > Nu_{\text{min,(iv)}} > Nu_{\text{min,(v)}} > Nu_{\text{min,(v)}}$.

For $\rho(p,T)$, $C_p(T)$, $\rho(T)$ and $k(T)$ variations, net $Nu$ increases after minima. This is due to contradictory mechanism as follows: (i) the induced inward radial convection due to $\rho(T)$ variation, (ii) reduce in $T_w$ and $(T_w-T_m)$ due to $C_p(T)$ variation and (iii) role of $k(T)$ variation is to increase induced radially-outward flow due to $\rho(p,T)$ variation.

The variation in $Nu$ due to $\rho(p,T)$, $\rho(p,T)$ and $C_p(T)$ variations and all properties variations shows an upward trend and seem to approach constant properties solution. It is found that for increase in $L/D$ from 100 to 125, an upward trend remains same and change in $Nu_{\text{exit}}$ is within 0.9% for above three cases. Table 2 shows $Nu_{\text{exit}}$
Table 2
The \( \text{Nu} \) and \( |\Delta \text{Nu}_{\text{CP}}| \) due to \( \rho(p,T), C_p(T), \mu(T) \) and \( k(T) \) variations with different combination of \( q_m^e, \ u_{\text{in,m}} \) and \( T_{\text{in},m} \).

| S. No. | \( q_m^e \) (W/cm\(^2\)) | \( u_{\text{in,m}} \) (m/s) | \( T_{\text{in},m} \) (K) | \( \text{Nu}_{\text{exit}} \) | \( |\Delta \text{Nu}_{\text{CP}}| \) (%) |
|--------|-------------------|-----------------|----------------|-------------|----------------|
| 1      | 7.5               | 15              | 278            | 4.377       | 0.32           |
| 2      | 0.1               | 20              | 4.359          | 0.01        |
| 3      | 2.5               | 25              | 4.347          | 0.38        |
| 4      | 5                 | 20              | 4.359          | 0.01        |
| 5      | 2.5               | 10              | 4.388          | 0.55        |
| 6      | 5                 | 10              | 4.359          | 0.11        |

The temperature profile remains flat across \( L_1 \), since, there is no heat transfer. Constant wall heat flux (5 W/cm\(^2\)) is applied across \( L_2 \) to develop temperature profile. The inlet and outlet boundary conditions are similar as described in case 1. For invariant gas properties the flow asymptotes to thermally and hydrodynamically fully-developed at \( L/D = 100 \) and \( \text{Nu}_{\text{CP}} = \frac{48}{\pi} \).

### 3.2. Numerical solution and physical mechanism due to variation in properties

For all properties assumed as constant, the velocity field remains same throughout computational domain. This field is already known and independent of temperature substituted in energy equation to obtain partial differential equations (PDEs). Therefore, momentum equation is uncoupled with thermal development of flow. The PDEs can be solved using variable separable method and has an analytical solution. However, when property variation is involved velocity field is coupled with other (temperature, momentum, energy, etc.) fields. This coupling is further enhanced by induced radial flow, hence, it is difficult to obtain an analytical solution. Therefore, it implies that energy equation remains unsolved which is attempted to solve by numerical solution. Jeong and Jeong [21] have studied Graetz problem with slip flow boundary conditions including effect of rarefaction, streamwise conduction and viscous dissipation. Barron et al. [22] modified Graetz problem through slip flow and temperature jump boundary condition and observed that slip flow at boundary augments heat transfer. Several researchers [23,24] work on Graetz problem referred in the literature as extended Graetz problem.

The prime objectives of discussing Graetz problem for variable fluid properties are as follows: (i) to validate importance of gas density sensitivity with respect to pressure and temperature, (ii) to explain coupling between temperature and velocity field. A numerical simulation is done with uniform velocity and temperature to understand the role of property variation coupled with thermal entrance effect.

**Fig. 7** illustrates variation in \( \text{Nu} \) for \( \rho(p,T), C_p(T), \mu(T) \) and \( k(T) \) variations due to \( u_{\text{in,m}} = 10 \) m/s and \( q_m^e = 7.5 \) W/cm\(^2\). The gas law has following form, \( p = \rho_{\text{atm}} + \rho_{\text{gauge}} / \gamma T \) where, \( \rho_{\text{atm}} \) is atmospheric value, i.e. operating pressure and local relative (gauge) pressure is zero. Therefore, for incompressible flow, the effect of

### Table 3

The \( \text{Nu}_{\text{in}} \) and \( \text{Nu}_{\text{exit}} \) due to different combination of properties and their variations.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Nu for combination of properties variations</th>
<th>( L/D ) and ( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( D = 750 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( L/D = 100 )</td>
</tr>
<tr>
<td>1</td>
<td>Constant properties ( \text{Nu}_{\text{exit}} )</td>
<td>4.365</td>
</tr>
<tr>
<td>2</td>
<td>( \rho(p,T), C_p(T), \mu(T) ) ( \text{Nu}_{\text{in}} )</td>
<td>4.278</td>
</tr>
<tr>
<td>3</td>
<td>( \rho(p,T), C_p(T) ) ( \text{Nu}_{\text{exit}} )</td>
<td>4.359</td>
</tr>
<tr>
<td>4</td>
<td>( \rho(p,T) ) ( \text{Nu}_{\text{min}} )</td>
<td>4.153</td>
</tr>
<tr>
<td>5</td>
<td>( \rho(p,T) ) ( \text{Nu}_{\text{exit}} )</td>
<td>4.181</td>
</tr>
<tr>
<td>6</td>
<td>( \rho(p,T) ) only ( \text{Nu}_{\text{exit}} )</td>
<td>4.514</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.563</td>
</tr>
</tbody>
</table>

(a) For \( u_{\text{in,m}} = 20 \) m/s and \( q_m^e = 7.5 \) W/cm\(^2\) with entrance effect

(b) For \( u_{\text{in,m}} = 20 \) m/s and \( q_m^e = 7.5 \) W/cm\(^2\) with Graetz problem

| S. No. | Fluid properties | \( \text{Nu}_{\text{in}} \) | \( |\Delta \text{Nu}_{\text{CP}}| \) (%) | \( \text{Nu}_{\text{exit}} \) | \( |\Delta \text{Nu}_{\text{CP}}| \) (%) |
|--------|----------------|----------------|-------------|----------------|-------------|
| 1      | \( \rho(p,T) \) | 4.076          | 6.59        | 4.129          | 1.59        |
| 2      | \( \mu(T) \) | 4.204          | 3.66        | 4.226          | 0.87        |
| 3      | \( k(T) \) | 4.444          | 1.84        | 4.444          | 0.41        |
| 4      | \( \rho(p,T), \mu(T) \) | 3.771          | 13.58       | 3.851          | 3.53        |
| 5      | \( C_p(T) \) and \( k(T) \) | 4.478          | 2.62        | 4.478          | 0.59        |
| 6      | \( \rho(p,T), C_p(T), \mu(T) \) and \( k(T) \) | 4.273          | 2.08        | 4.351          | 0.48        |
density variation is more through its temperature than pressure dependence as \( M < 0.3 \). The \( \rho(p,T) \) variation effects on \( \text{Nu} \) are indicative in micro-convection, due to significant effect of \( s_{pp} \) resulting from steep temperature gradients. The \( s_{pp} \) is insignificant, therefore, the deviation of \( \text{Nu} \) between ideal and incompressible ideal gas approximation is negligible. A comparison of \( \text{Nu} \) due to \( q_{\text{w}}^* = 5 \text{ W/cm}^2 \) for combination of properties variations is shown in Table 3(b), where minimum \( \text{Nu} \) is differing and

\[
|\Delta \text{Nu}| = \left| \left( \frac{\text{Nu}_{\text{min}} - \text{Nu}_{\text{CP}}}{\text{Nu}_{\text{CP}}} \right) \right| \times 100\%.
\]

It is observed that for \( \rho(p,T), C_p(T), \mu(T) \) and \( k(T) \) variations, \( \text{Nu} \) decreases compared to constant properties solution. The reason for this is explained by various mechanisms due to each property variation as illustrated in Table 4.

This can be jointly explained as follows:

(i) Effect of \( \rho(p,T) \) variation is to induce inward radial velocity due to sharpening of \( u(r,z) \) and \( T(r,z) \) profile, thereby reduces \( \text{Nu} \). Inward radial flow is in the direction of diffusion of wall boundary condition which is counteracted by outward radial velocity. At \( z/D = 80 \) and onwards increasing radial velocity flattens \( u(r) \) profile, hence, flattening effect increases \( \text{Nu} \).

(ii) Effect of \( C_p(T) \) also becomes relevant due to axial temperature gradients and increases \( \text{Nu} \) towards the end of micro-tube because of reducing \( \partial T_{\text{m}}/\partial z \) and \( \left( T_w - T_m \right) \).

(iii) Effect of \( \mu(T) \) variation leads to non-negligible radial convection relative to axial convection and induces inward radial flow by sharpening \( u(r) \) profile, thereby reduces \( \text{Nu} \).

(iv) Effect of \( k(T) \) variation is to decrease sharpening of temperature profile and induces non-negligible axial conduction, thereby increases \( \text{Nu} \). As the gradients are gradually developed, outward radial velocity causes decrease in \( \text{Nu} \) but flattening of temperature profile increases \( \text{Nu} \).

### 4. Conclusions

(i) Due to sharpening of \( u(r,z) \) and \( T(r,z) \) profile, resulting from radially-inward flow and contradicting mechanisms involved with each property variation, there occurs a minimum \( \text{Nu} \). The increase in \( \text{Nu} \) after the minima is a result of dominance of \( k(T) \) and \( C_p(T) \) variations and reducing \( \text{Nu} \) minima is due to \( \mu(T) \) variation.

(ii) The effect of \( \rho(p,T) \) and \( k(T) \) variations is to induce outward radial velocity which decreases the inward radial flow, hence, rate of decrease of \( \text{Nu} \) is alleviated and \( \text{Nu} \) minima occurs at higher value.

(iii) Gases have lower \( C_p \) than liquid which increases axial temperature gradients and the role of property variation along the flow. The role of \( C_p(T) \) variation is to reduce \( \partial T_{\text{m}}/\partial z \), \( T_w \), and \( \left( T_w - T_m \right) \), thereby augmenting \( \text{Nu} \). Reduction in \( T_w \) is advantageous for cooling of the wall.

---

**Table 4**

<table>
<thead>
<tr>
<th>Thermophysical properties</th>
<th>Effect on Velocity field</th>
<th>Temperature field</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density (( \rho ))</td>
<td>Direct effect. Flattens axial profile (significant)</td>
<td>Direct effect (significant)</td>
</tr>
<tr>
<td>Specific heat (( C_p ))</td>
<td>Indirect effect (insignificant)</td>
<td>Direct effect. Reduces ( T_{w-m} ) at high temperature (significant)</td>
</tr>
<tr>
<td>Viscosity (( \mu ))</td>
<td>Direct effect. Sharpens axial profile (significant)</td>
<td>Indirect effect (insignificant)</td>
</tr>
<tr>
<td>Thermal conductivity (( k ))</td>
<td>Indirect effect. Flattens axial profile (significant)</td>
<td>Direct effect. Flattens temperature profile (significant)</td>
</tr>
</tbody>
</table>

**Fig. 7**. Variation of \( \text{Nu} \) versus \( z/D \) due to combination of \( \rho(p,T), C_p(T), \mu(T) \) and \( k(T) \) variations for \( u_{\text{m in}} = 10 \text{ m/s} \) and \( q_{\text{w}}^* = 5 \text{ W/cm}^2 \) with Graetz problem.
(iv) Graetz problem shows gas density variation with thermal entrance effect at low Mach number due to significance of $s_{PT}$ and steep temperature gradients.

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