Abstract—This paper presents a method for achieving quasi-sliding mode for uncertain systems using a fast output sampling control strategy that avoids switching of control and, hence, avoids chattering. This method does not need the system states for feedback as it makes use of only the output samples for designing the controller. Thus, this methodology is more practical and easy to implement. The design technique is illustrated through two numerical examples.

Index Terms—Discrete-time systems, output feedback, uncertain systems, variable structure systems.

I. INTRODUCTION

THE sliding-mode control (SMC) theory is based on the concept of varying the structure of the controller based on the changing state of the system in order to obtain a desired response [1], [2]. A switching control action is used to switch between different structures, and the system state is forced to move along a chosen manifold called the switching manifold, which determines the closed-loop-system behavior [3], [4]. In recent years, considerable efforts have been put in the study of the concepts of digital sliding-mode (DSM) controller design [5]–[8]. In case of the DSM design, the control input is applicable only at certain sampling instants, and the control effort is constant over the entire sampling period. Moreover, when the states reach the switching surface, the subsequent control would be unable to keep the states confined to the surface. As a result, DSM can undergo only quasi-sliding mode, i.e., the system states would approach the sliding surface but would generally be unable to stay on it. Thus, in general, DSM does not possess the invariance property found in continuous-time sliding mode.

Bartoszewicz [6] proposed a state-feedback-based control law for uncertain systems that guarantees discrete sliding mode. Moreover, this law avoids the switching function present in other SMC algorithms [8], thus avoiding chatter. However, these SMC strategies require full-state feedback. But, in practice, all the states of the system are not always available for measurement. Since the output is available for measurement, output feedback can be used for controller design. Few research works are available, which deal with SMC design using output feedback [9]–[11]. The problem of static output feedback has been studied in considerable details. How-

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appropriate dimensions with \((\Phi_\tau, \Gamma_\tau)\) being controllable and \((\Phi_\tau, C)\) being observable.

Let the desired sliding manifold be
\[ s(k) = c^T x(k) = 0 \]
governed by the parameter vector \(c^T\) such that \(c^T\Gamma_\tau \neq 0\), and the resulting quasi-sliding motion is stable. Let the disturbance be bounded such that \(d(k) = c^T \tilde{d}(k)\) satisfies the inequality
\[ d_1 \leq d(k) \leq d_u \]
where \(d_1\) and \(d_u\) are known upper and lower bounds, respectively. Define the terms \(d_0 = 0.5(d_1 + d_u)\), \(\delta_d = 0.5(d_u - d_1)\), and \(s(k) = c^T x(k)\). The quasi-sliding mode is defined as the motion such that \(|s(k)| \leq \varepsilon\), where the positive constant \(\varepsilon\) is called the quasi-sliding-mode bandwidth.

Bartoszewicz proposed a new reaching law
\[ s(k + 1) = d(k) - d_0 + s_d(k + 1) \tag{2} \]
where \(s_d(k)\) is an \textit{a priori} known function such that the following apply:

1) if \(|s(0)| > 2\delta_d\), then
\[ s_d(0) = s(0) \]
\[ s_d(k) \cdot s_d(0) \geq 0, \quad \text{for any } k \geq 0 \]
\[ s_d(k) = 0, \quad \text{for any } k \geq k^* \]
\[ |s_d(k + 1)| < |s_d(k)| - 2\delta_d, \quad \text{for any } k < k^* \tag{3} \]

2) otherwise, i.e., if \(|s(0)| \leq 2\delta_d\), then \(s_d(k) = 0\) for any \(k \geq 0\).

The positive integer \(k^*\) is chosen by the designer to achieve a good tradeoff between faster convergence and magnitude of the control \(u\).

One possible definition of \(s_d(k)\) when \(|s(0)| > 2\delta_d\) is
\[ s_d(k) = \frac{k^* - k}{k^*} s(0), \quad k = 0, 1, \ldots, k^* \]
\[ k^* < \frac{|s(0)|}{2\delta_d}. \]
The control law that satisfies the reaching law in (2) can be computed by using (1) as
\[ u(k) = -(c^T\Gamma_\tau)^{-1}(c^T\Phi_\tau x(k) + d_0 - s_d(k + 1)) \]
The control law so designed guarantees that for any \(k \geq k^*\), the system states satisfy the inequality
\[ |s(k)| = |d(k - 1) - d_0| \leq \delta_d. \]

Hence, the states of the system settle within a quasi-sliding mode band whose width is less than half of the width of the band in [8].

### III. State and Output Relation for Multirate Uncertain Systems

Consider the system in (1) sampled with sampling interval \(\Delta = \tau/N\), where \(N\) is chosen to be an integer greater than the observability index of the system. Let this system be represented by the matrices \(\Phi, \Gamma,\) and \(C\). Let us assume that the disturbance vector present in the \(\tau\) system in (1) manifests itself in the \(\Delta\) system dynamics as
\[ x(k\tau + (j + 1)\Delta) = \Phi x(k\tau + j\Delta) + \Gamma u(k\tau) + d'(k) \]
\[ y(k\tau + j\Delta) = C x(k\tau + j\Delta), \quad j = 0, 1, \ldots, N - 1 \tag{4} \]
where \(d'(k)\) is the disturbance in the \(\Delta\) system due to \(\tilde{d}(k)\). Using (4), the state vector at \(j = 0, 1, \ldots, N - 1\) can be computed in terms of \(x(k\tau)\) as [15]
\[ x(k\tau + \Delta) = \Phi x(k\tau) + \Gamma u(k\tau) + d'(k) \]
\[ x(k\tau + 2\Delta) = \Phi x(k\tau + \Delta) + \Gamma u(k\tau) + d'(k) \]
\[ = \Phi^2 x(k\tau) + (\Phi \Gamma + \Gamma) u(k\tau) + (\Phi + I)d'(k) \]
\[ \vdots \]
\[ x((k + 1)\tau - \Delta) = \Phi^{N - 1} x(k\tau) + \left(\sum_{i=0}^{N-2} \Phi^i \Gamma\right) u(k\tau) + \left(\sum_{i=0}^{N-2} \Phi^i\right) d'(k) \]
\[ + \left(\sum_{i=0}^{N-1} \Phi^i\right) d'(k). \tag{5} \]

Using the properties of discrete-time linear time-invariant (LTI) systems and comparing (5) with (1), we arrive at the relationship
\[ d'(k) = \left(\sum_{i=0}^{N-1} \Phi^i\right)^{-1} \tilde{d}(k). \]

Now, if the system output is sampled after every \(\Delta\) sec and input is applied at every \(\tau\) sec, then, we have the relation between the system states \(x(k)\) and the lifted output \(\tilde{y}_{k+1}\) as
\[ x((k + 1)\tau) = \Phi_\tau x(k\tau) + \Gamma_\tau u(k\tau) + \tilde{d}(k) \tag{6} \]
\[ \tilde{y}_{k+1} = C_\tau x(k) + D_0 u(k) + C_d \tilde{d}(k) \tag{7} \]
where
\[ y_k = [y((k-1)\tau) \quad y((k-1)\tau + \Delta) \quad \cdots \quad y(k\tau - \Delta)]^T \]
\[ C_0 = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^{N-1} \end{bmatrix} \]
\[ D_0 = \begin{bmatrix} 0 \\ CT \\ C(\Phi\Gamma + \Gamma) \\ \vdots \\ C\sum_{i=0}^{N-2}\Phi^i \end{bmatrix} \]
\[ C_d = \begin{bmatrix} 0 \\ C\Phi + C \\ \vdots \\ C\sum_{i=0}^{N-2}\Phi^i \end{bmatrix}^{-1} \]
Thus, we have the relation
\[ C_0^Ty_{k+1} = C_0^T(C_0x(k) + D_0u(k) + C_d\tilde{d}(k)). \]
If \( N \) is chosen greater than the observability index of the system, then the matrix \( C_0 \) would be of rank \( n \) (by the definition of observability index). Hence, \( C_0^T C_0 \) would be an \( n \times n \) matrix of rank \( n \). Hence, as the system is assumed to be observable, \( (C_0^T C_0)^{-1} \) exists. Hence, using (6) and (7), the state \( x(k+1) \) can be deduced to be
\[ x(k) = (C_0^T C_0)^{-1} C_0^T (y_{k+1} - D_0u(k) - C_d\tilde{d}(k)) \]
\[ x(k+1) = L_y y_{k+1} + L_u u(k) + L_d \tilde{d}(k) \quad (8) \]
where
\[ L_y = \Phi_r (C_0^T C_0)^{-1} C_0^T \]
\[ L_u = \Gamma_r - \Phi_r (C_0^T C_0)^{-1} C_0^T D_0 \]
\[ L_d = I - \Phi_r (C_0^T C_0)^{-1} C_0^T C_d. \]
Thus, the state \( x(k) \) can be expressed using the lifted output vector \( y_k \) as
\[ x(k) = L_y y_k + L_u u(k-1) + L_d \tilde{d}(k-1). \quad (9) \]

**IV. Proposed Algorithm**

Consider the system described by (6) and (7). We define a new variable \( e(k) \) as
\[ e(k) = e^T \Phi_r L_d \tilde{d}(k). \]
Since the disturbance is bounded, we have
\[ e_1 \leq e(k) \leq e_u. \]
Let us define the average value of \( e(k) \) and the maximum deviation of \( e(k) \) from this value as
\[ e_0 = 0.5(e_u + e_1), \quad \delta_e = 0.5(e_u - e_1) \]
respectively. Now, we propose a new reaching law for output-feedback sliding mode for the system in (1) as
\[ s(k+1) = d(k) - d_0 + e(k-1) - e_0 + s_d(k+1) \]
where \( s_d(k+1) \) is an a priori known function, which satisfies the conditions in (3) for the bound of \( (\delta_d + \delta_e) \) instead of \( \delta_d \). We consider \( s_d(k) \) to be of the form
\[ s_d(k) = \frac{k^* - k}{k^*} s(0), \quad k = 0, 1, \ldots, k^*; \]
\[ k^* < \frac{|s(0)|}{2(\delta_d + \delta_e)}; \quad s_d(k) = 0; \quad k > k^*. \]
Thus
\[ c^T (\Phi_r x(k) + \Gamma_r u(k) + \tilde{d}(k)) = d(k) - d_0 + e(k-1) - e_0 + s_d(k+1). \]
\[ c^T \Phi_r x(k) + c^T \Gamma_r u(k) + d(k) = d(k) - d_0 + e(k-1) - e_0 + s_d(k+1) \]
\[ c^T \Phi_r x(k) + c^T \Gamma_r u(k) = -d_0 + e(k-1) - e_0 + s_d(k+1). \]

Now, the control input can be derived as
\[ u(k) = -(c^T \Gamma_r)^{-1} (c^T \Phi_r x(k) + d_0 - e(k-1) + e_0 - s_d(k+1)). \quad (10) \]
Using (9)
\[ u(k) = -(c^T \Gamma_r)^{-1} \left[ c^T \Phi_r [L_y y_k + L_u u(k-1) + L_d \tilde{d}(k-1)] \right] + d_0 - e(k-1) + e_0 - s_d(k+1) \]
\[ = -(c^T \Gamma_r)^{-1} \left[ c^T \Phi_r L_y y_k + c^T \Phi_r L_u u(k-1) + e(k-1) - e(k-1) + d_0 + e_0 - s_d(k+1) \right] \]
\[ = -(c^T \Gamma_r)^{-1} \left[ c^T \Phi_r L_y y_k + c^T \Phi_r L_u u(k-1) + d_0 + e_0 - s_d(k+1) \right]. \quad (11) \]
Hence, the control input can be computed using the past output samples and the immediate past input signal. But, at \( k = 0 \), there are no past outputs for use in control, hence, \( u(0) \) is obtained by ignoring \( e(k-1) \) and \( e_0 \) (as we expect no disturbance before \( k = 0 \) to affect the system) from (10) and assuming an initial state \( x(0) \) to obtain
\[ u(0) = -(c^T \Gamma_r)^{-1} \left( c^T \Phi_r x(0) + d_0 - s_d(1) \right). \]
When the control input derived from (11) is applied to the system, the system obeys the reaching law

\[
\begin{align*}
    s(k+1) &= d(k) - d_0 + e(k-1) - e_0 + s_d(k+1) \\
    s(k) &= d(k-1) - d_0 + e(k-2) - e_0 + s_d(k)
\end{align*}
\]

for \( k > \max(k^*, 2) \), \( s_d(k) = 0 \), and therefore

\[
\begin{align*}
    s(k) &= d(k-1) - d_0 + e(k-2) - e_0 \\
    |s(k)| &= |d(k-1) - d_0 + e(k-2) - e_0| \\
    &\leq |d(k-1) - d_0| + |e(k-2) - e_0| \\
    &= \delta_d + \delta_e \\
    |s(k)| &\leq \delta_d + \delta_e.
\end{align*}
\]

\[\Box\]

V. NUMERICAL EXAMPLE

A. Example 1

Consider the system cited in [6]

\[
\begin{align*}
    x(k+1) &= \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
    y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k).
\end{align*}
\]

The sliding line is chosen as \( e^T = [1 \ 1] \). Computation of the parameters gives

\[
\begin{align*}
    L_y &= \begin{bmatrix} -0.707 & 0.707 \\ -0.854 & 0.854 \end{bmatrix} \quad L_u = \begin{bmatrix} 0.293 \\ 1.146 \end{bmatrix} \quad L_d = \begin{bmatrix} -0.707 & 0 \\ -0.854 & 1 \end{bmatrix} \\
    d_u &= d_1 = 1 \quad e_u = e_1 = 1.5 \quad \delta_d = \delta_e = 0 \quad k^* = 15.
\end{align*}
\]

For an initial condition \( x(0) = [1000 \ 0]^T \), using (11), the SMC input is derived to be

\[
\begin{align*}
    u(k) &= [1.987 - 2.987] y_k - 2.013 u(k-1) + s_d(k+1) - 2.5
\end{align*}
\]

The simulation results are shown in Figs. 1 and 2. Fig. 1 shows the evolution of the variable \( s(k) \). It can be seen that the state trajectory converges to the sliding line in finite time after \( k^* = 15 \) sampling instants. Also, there is a quasi-sliding mode motion without chatter. The output response of the system is given in Fig. 2. It is evident here that the output converges without exhibiting undesirable chatter. The system responses are exactly the same as obtained through state feedback in [6].

B. Example 2

Consider the discrete-time LTI system

\[
\begin{align*}
    x(k+1) &= \begin{bmatrix} 0 & 1 \\ 0.4 & -0.3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\
    &\quad + \begin{bmatrix} \sin(k/2) \exp(-k/5) \end{bmatrix} \\
    y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k).
\end{align*}
\]

For an initial condition \( x(0) = [10 \ 0]^T \), using (11), the SMC input is derived to be

\[
\begin{align*}
    u(k) &= \begin{bmatrix} -3.019 & 4.233 \end{bmatrix} y_k - 1.017 u(k-1) + s_d(k+1) - 0.06
\end{align*}
\]

The simulation results for a time-varying disturbance are shown in Figs. 3 and 4. Fig. 3 shows the evolution of the sliding function \( s(k) \). The plot compares the performance of the proposed controller with the state-feedback-based control
There is a possibility that in some of the cases, the disturbance vector can be such that \( d(k) \) and \( e(k-1) \) are of opposite sense (as in the case of Example 2). In such a case, the proposed output-feedback-based control would perform better than a state-feedback-based control. However, this cannot be generalized. In most cases, the state-feedback-based algorithm would give a better performance.

If this algorithm is applied to deterministic systems or systems with a constant disturbance, i.e., \( \delta_d = \delta_e = 0 \), by choosing the value of \( k^* \) to be any suitable positive integer, then the system states would converge exactly to the sliding surface after \( k^* \) time samples without any chatter and effectively reducing the quasi-sliding mode band to zero. In this case, both the state-feedback-based control and the output-feedback-based control systems behave in the same manner.

VI. Conclusion

A new control technique is developed for discrete-time SMC combining the Bartoszewicz’s reaching law approach and a modified form of FOS technique. This control algorithm avoids the use of a switching input; hence, the system response does not have undesirable chatter. Moreover, since the control law uses only the past output samples and past control input instead of the system states, it is more practical. However, the quasi-sliding mode band is increased when compared with the control algorithm presented in [6]. The proposed algorithm was verified and compared with the algorithm presented in [6] through two numerical examples.

The proposed technique has been applied to vibration control of smart-structure beam, and its implementation to the same is under progress.

REFERENCES


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