Hybrid-interface finite element for laminated composite and sandwich beams

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Abstract

A novel, 9-node, two-dimensional hybrid-interface finite element has been presented to analyze laminated composite beams based on the principle of minimum potential energy. Fundamental elasticity relationship between the components of stress, strain and displacement fields are maintained throughout the elastic continuum as the transverse stress components have been invoked as nodal degrees of freedom (DOF). Continuity of the transverse stresses at lamina interface is maintained, as the transverse stress components are treated as DOF along-with displacement DOF at interface nodes. Each lamina is modeled using hybrid-interface elements at the top and bottom lamina interfaces and conventional displacement-based 9-node elements sandwiched between these interface elements. Results obtained from the present model have been found to be in excellent agreement with the elasticity solutions for thin and thick laminated composite as well as sandwich beams. Present formulation can be used effectively to combine hybrid-interface elements having transverse stress and displacement components as nodal DOF with conventional 9-node displacement elements for realistic estimate of the transverse stresses.

Keywords: Laminated composite beams; Sandwich beams; Mixed formulation; Hybrid-interface element

1. Introduction

Use of fiber reinforced composite laminates as primary load carrying components in various structures has increased in manifolds due to better stiffness-to-weight and strength-to-weight ratios than those of conventional materials. Laminated composites are characterized by the material which has material orthotropic properties in the laminate plane and low ratio of the transverse shear modulus to in-plane modulus. As a result, the transverse shear stress/deformation values have significant importance. For example, the interlaminar shear stress (local values) on free edge of an angle-ply laminate can cause edge delamination. Similarly, the interlaminar transverse normal stress (global values) may cause delamination in cross-ply laminates. Therefore, a formulation providing accurate solution of local values (interface transverse stresses) and global values (displacement components and in-plane stresses) is essential.

Many equivalent single layer (ESL) techniques have been developed for composite stress analysis. These can be classified as classical laminated plate theory (CLPT) [1–3], first order shear deformation theory (FSDT) [4–6] and higher order shear deformation theories (HSDT). Among these techniques, HSDT received most attention in recent years. Many HSDT, such as those presented by Lo et al. [7], Reddy [8] and Kant and Pandya [9], are available for analysis of composite laminates. However, interlaminar stresses cannot be predicted correctly by using such two-dimensional theories. To circumvent the problem, higher order ESL theories with post-processing techniques using equilibrium equations for stress recovery have been employed by Rolfes et al. [10], Noor et al. [11], Reddy [12] and Lo et al. [13]. Although such techniques may predict accurate solution, their use is generally tedious and not suitable for structures with complex geometry and loading.

Elasticity solutions for composite laminates (e.g. [14–16]) indicate that interlaminar continuity of the transverse normal and shear stresses as well as the layer-wise continuous displacement field through the thickness are essential requirements in analysis. Thus, a layer-wise analysis is often required for laminated composite structures even if it is computationally
According to historical review of ZZTs presented by Car- to alleviate the requirement of C1 continuity on transverse form of the FZZT and a generalized form of HZZT for beams zig-zag theories (HZZT). Averill proposed a generalized significant improvement to the FZZT called as higher order dent of the number of layers. Di Sciuva as well as others made as total number of degrees of freedom (DOF) were indepen-
thickness. FZZT proved to be a numerically efficient method piece-wise linear for each lamina and continuous through-the- 
permits both displacement and transverse stress assumptions. Reissner propo
sulted theories have been proposed by Reddy [17], Soldatos [18], Wu and Kuo [19], Carrera and Demasi [20–22], and others, providing satisfactory results for both the global values (e.g. deflections and flexural stresses) as well as the local values (e.g. transverse stresses) of thin and thick laminates. However, continuity of the transverse stress components cannot be ensured in these models. New class of laminate theory called zig-zag theories (ZZTs) was develop-
ed to fulfill interlaminar continuity of transverse stresses. According to historical review of ZZTs presented by Carrera [23]: (i) Lekhnitskii [24] was the first to propose a ZZT for multilayered plate using elasticity relations; (ii) Ambartsumyan [25] proposed a ZZT by extending the well-known Reissner–Mindlin theory to layered, anisotropic plates; and (iii) Reissner [26], proposed a ZZT using variational theorem that permits both displacement and transverse stress assumptions. Di Sciuva [27,28] also presented first order zig-zag theory (FZZT), wherein in-plane displacements were assumed to be piece-wise linear for each lamina and continuous through-the-
thickness. FZZT proved to be a numerically efficient method as total number of degrees of freedom (DOF) were independent of the number of layers. Di Sciuva as well as others made significant improvement to the FZZT called as higher order zig-zag theories (HZZT). Averill [29] proposed a generalized form of the FZZT and a generalized form of HZZT for beams to alleviate the requirement of C1 continuity on transverse deflection. A sublaminate zig-zag method which combines benefits of both discrete-layer-wise and HZZT was presented by Cho and Averill [30], a three-dimensional model zig-zag sublaminate technique was presented by Pantano and Averill [31]. Layer-wise model with continuity of transverse shear stresses was first presented by Ambartsumyan [32]. Toledano and Murakami [33] presented a layer-wise technique based on Reissner’s new mixed variational principle by taking transverse stresses to be quadratic function of local thickness. Lu and Liu [34] proposed a layer-wise theory, based on techniques by Di Sciuva [27,28], Hinrichsen and Palazzotto [35], Toledano and Murakami [33] and Reddy [12], to calculate shear stresses directly from constitutive equations instead of being recovered from equilibrium equation.

The hybrid-stress FE method by Mau et al. [36] and others satisfies the continuity condition explicitly. The interlaminar stresses can be obtained directly from the assumed stress function. Spilkar [37], for example, presented a hybrid-stress FE model based on Reissner’s mixed variational principle in order to reduce numerical complexity with a compromise in the quality of solution.

It has been assumed in the hybrid FE models developed using Reissner’s mixed variational principle (e.g. [20,21,38]) that stress fields are independent of displacement fields. Therefore fundamental elasticity relations cannot be satisfied and the continuity of some relevant variables is only guaranteed in weak sense. On the other hand, a mixed FE model based on displacement theory satisfying fundamental elasticity relations has been presented by Desai and Ramtekkar [39]. This model has been shown to provide reliable results for stress and displacement of laminated beams, satisfying continuity of displacements as well
as the transverse stresses through the thickness. Lee and Liu [40] presented a theory which satisfies the continuity of both interlaminar shear and normal stresses based on the principle of minimum potential energy. Present hybrid-interface formulation is aimed to provide an effective combination of conventional displacement elements with compatible hybrid elements at lamina interfaces without compromising continuity of the displacement and transverse stresses and quality of solution.

A layer-wise FE model with displacement and the transverse stress components as primary variables at interface nodes and only displacement components at non-interface nodes has been considered. This formulation satisfies requirement of the transverse stress continuity in addition to the continuity of displacement fields through the thickness of the laminates as shown in Fig. 1(a). The transverse stress components at interface are evaluated directly in the hybrid-interface FE model. Integration of equilibrium equations in the estimation of the transverse stresses, which involves differentiation of in-plane stresses and displacement fields leading to approximations in calculation, is not a requirement of the present formulation.

A two-dimensional, 9-node hybrid-interface FE has been developed in the present work by using the minimum potential energy principle satisfying the entire requirement discussed above for an N-layered laminated beam. The transverse stress quantities ($\sigma_z$ and $\tau_{xz}$) have been taken as DOF at interface nodes in addition to displacement components ($u$ and $w$) to ensure through thickness continuity of these terms. Each lamina is discretized using a row of hybrid-interface elements at the top and the bottom interface and with rows of conventional 9-node elements in the continuum in-between. Compatibility of hybrid elements at interfaces with conventional displacement elements is ensured.

2. Theoretical formulation

An anisotropic laminated composite beam consisting of N-layers of orthotropic lamina shown in Fig. 1(b) has been considered. The beam has been discretized using a combination of 9-node hybrid-interface elements and conventional 9-node displacement elements as shown in Fig. 2(a).
2.1. Kinematics

A 9-node hybrid-interface FE model for bottom interface level shown in Fig. 2(b) has been developed by considering displacement fields \( u(x, z) \) and \( w(x, z) \) having quadratic variation along the longitudinal axis \( x \) and cubic variation along the transverse axis \( z \). Displacement fields can be expressed as

\[
u(x, z) = \sum_{i=1}^{3} g_i a_{0i} + z \sum_{i=1}^{3} g_i a_{1i} + z^2 \sum_{i=1}^{3} g_i a_{2i} + z^3 \sum_{i=1}^{3} g_i a_{3i},
\]

\[w(x, z) = \sum_{i=1}^{3} g_i b_{0i} + z \sum_{i=1}^{3} g_i b_{1i} + z^2 \sum_{i=1}^{3} g_i b_{2i} + z^3 \sum_{i=1}^{3} g_i b_{3i},\]

where quadratic shape function in \( \xi \)-direction are

\[g_1 = \frac{\xi}{2}(\xi - 1), \quad g_2 = 1 - \xi^2, \quad g_3 = \frac{\xi}{2}(1 + \xi),\]

\[\xi = x/L_x.\]

Further, the generalized coordinates \( a_{mi} \) and \( b_{mi} \) \((m = 0, 1, 2, 3; i = 1, 2, 3)\) are functions of element coordinate axis ‘\( z \)’. The element’s coordinate axes \( x, z \) are parallel to the laminate coordinates \( X, Z \).

Variation of the displacement fields has been assumed to be cubic along the thickness of element although there are only three nodes along \( z \)-axis of an element (Fig. 2(a)). Such a variation is required for invoking the transverse stress components \( \tau_{xz} \) and \( \sigma_z \) as the nodal DOF at interface nodes in the present formulation. Further, it also ensures parabolic variation of the transverse stresses through thickness of an element.

2.2. Constitutive equations

Each lamina in the laminate has been considered to be in the state of plane-strain in the \( X-Z \) plane. Constitutive relation for a typical \( i \)th lamina with reference to the coordinate system can presented as

\[\sigma = [D][\epsilon],\]
where

\[ \{\sigma\} = [\sigma_x \sigma_z \tau_{xz}]^T, \]  
(4b)

\[ [D] = \begin{bmatrix} D_{11} & D_{13} & 0 \\ D_{13} & D_{33} & 0 \\ 0 & 0 & D_{55} \end{bmatrix}, \]  
(4c)

\[ \{v\} = [e_x e_z \gamma_{xz}]^T. \]  
(4d)

Coefficients \(D_{mn}\) in Eq. (4c) are the elastic constants.

2.3. FE formulation

The transverse stresses can be related to displacement derives (strains) from the constitutive Eqs. (4) as

\[ \frac{\partial w}{\partial z} = \frac{1}{D_{33}} \left[ \sigma_z - D_{13} \frac{\partial u}{\partial x} \right], \]  
(5)

\[ \frac{\partial u}{\partial z} = \left( \frac{\tau_{xz}}{D_{55}} - \frac{\partial w}{\partial x} \right). \]  
(6)

Substituting Eqs. (1), (2), (5) and (6) at the bottom nodes of an element, following expressions for the displacement fields \(u(x, z)\) and \(w(x, z)\) can be obtained:

\[ u(x, z) = \sum_{n=1}^{9} g_1 f_q u_n + \sum_{n=1}^{3} g_1 f_p \frac{\partial u_n}{\partial z}, \]  
(7)

\[ w(x, z) = \sum_{n=1}^{9} g_2 f_q w_n + \sum_{n=1}^{3} g_4 f_p \frac{\partial w_n}{\partial z}. \]  
(8)

Here \(n\) is the node number (1, 2, \ldots, 9) of element shown in Fig. 1c. On the other hand, \(u_n\) and \(w_n\) are the nodal displacement DOF and \(\partial u_n/\partial z\) and \(\partial w_n/\partial z\) contains the nodal transverse stress DOF at the interface nodes. Further, subscript \(i\) represents the position of node with respect to \(\xi\) coordinate i.e. \(i = 1, 2\) and 3 for the nodes with \(\xi = -1, 0, 1\), respectively. Subscript \(q\) represents position of node with respect to \(\eta(\eta = z/L_z)\) coordinate i.e. \(q = 1, 2\) and 3 for nodes with \(\eta = -1, 0, 1\), respectively. Further, \(p = 4\) for nodes with mixed DOF i.e. at \(\eta = -1\) for the bottom interface element.

Cubic shape functions for the \(\eta\)-direction are

\[ f_1 = \frac{1}{4}(-8\eta + 2\eta^2 + 6\eta^3), \quad f_2 = \frac{1}{4}(4 + 8\eta - 4\eta^2 - 8\eta^3), \]

\[ f_3 = \frac{1}{4}(2\eta^2 + 2\eta^3), \quad f_4 = \frac{h}{4}(-4\eta + 4\eta^3). \]  
(9)

Eqs. (7) and (8) can be written in the standard finite element form as

\[ \{u\} = [u \ w]^T = \{N\}[d], \]  
(10a)

where

\[
[N] = \begin{bmatrix}
N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_9
\end{bmatrix},
\]  
(10b)

\[
[N_n]_{(2 \times 4)}, \quad \text{for } n = 1, 2 \text{ and } 3 \text{ and } [N_n]_{(2 \times 2)}, \quad \text{for } n = 4 \text{ to } 9
\]  
(10c)

\[ \{d\} = [d_1 \ d_2 \ d_3 \ d_4 \ d_5 \ d_6 \ d_7 \ d_8 \ d_9], \]  
(10d)

\[ \{d_n\}^T = [u_n \ w_n] \quad \text{for } n = 1 \text{ to } 3, \]

\[ \{d_n\}^T = [u_n \ w_n] \quad \text{for } n = 4 \text{ to } 9. \]  
(10e)

Further,

\[
[N] = \begin{bmatrix}
g_{f_q} f_q & -g_{f_p} f_q & 1 & 0 \\
g_{f_p} f_q & g_{f_p} f_q & 0 & 1 \\
0 & g_{f_p} f_q & 1 & 0 \\
\end{bmatrix}
\]  
(11a)

\[
[N] = \begin{bmatrix}
g_{f_q} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 \\
\end{bmatrix}
\]  
(11b)

and

\[ g_i' = \frac{\partial g_i}{\partial z}. \]  
(12)

The total potential energy \(\Pi\) of the laminated beam having width ‘\(b\)’ can be obtained from

\[
\Pi = \frac{1}{2} \int_A \{\varepsilon\}^T \{\sigma\} \ dx \ dz - \int_A \{u\}^T \{p_b\} \ dx \ dz - \int \{u\}^T \{p_t\} \ dx
\]

\[ \quad - \sum_{i=1}^{N_p} \{u\}^T \{P\}_i, \]  
(13)

where \(\{p_b\}\) represents the body force vector per unit volume, \(\{p_t\}\) represents the traction load vector acting on an edge, \(N_p\) is the number of point loads and \(\{P\}_i\) represents the point load vector acting on the laminated beam.

The strain vector \(\{\varepsilon\}\) and the stress vector \(\{\sigma\}\) can be expressed as

\[
\{\varepsilon\} = [B]\{d\}, \]  
(14)

and

\[
\{\sigma\} = [D][B]\{d\}. \]  
(15)

Here

\[
[B] = \begin{bmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 & B_9
\end{bmatrix}
\]  
(16a)

with

\[
[B] = \begin{bmatrix}
g_{f_q} f_q & -g_{f_p} f_q & 1 & 0 \\
g_{f_p} f_q & g_{f_p} f_q & 0 & 1 \\
0 & g_{f_p} f_q & 1 & 0 \\
\end{bmatrix}
\]  
(16b)
The global equation can then be obtained by assembly as

\[ [B]_n = \begin{bmatrix} g_i f_q & 0 \\ 0 & g_i f_i \\ g_i f_q & g_i f_q \end{bmatrix}, \quad n = 4 \text{ to } 9. \quad (16c) \]

Further,

\[ g_i = \frac{\partial^2 g}{\partial x^2} \quad \text{and} \quad f_j = \frac{\partial f}{\partial x}, \quad j = q \text{ or } p. \quad (17) \]

Minimization of the total potential energy functional expressed in Eq. (13) yields the element property matrix \([K]\) and the element influence vector \([f]\) as

\[ [K] = b \int_{-L_z}^{L_z} [B]^T[D][B] \, dx \, dz, \quad (18) \]

\[ [f] = b \int_{-L_z}^{L_z} \int_{-L_z}^{L_z} [N]^T[p_b] \, dx \, dz + \sum_{i=1}^{N_p} [N]^T_i \{P\}_i, \quad (19) \]

The global equation can then be obtained by assembly as

\[ [K][Q] = [F], \quad (20) \]

where \([K],[Q]\) and \([F]\) are, respectively, the global property matrix, the global DOF vector and the global influence vector.

### 3. Numerical results and discussion

A computer program incorporating the present two-dimensional 9-node hybrid-interface elements with conventional 9-node displacement elements has been developed in FORTRAN-90 for analysis of symmetric/unsymmetric crossply laminated and sandwich beams. Numerical computation for various examples has been performed. Results have been compared with the available elasticity and finite element solutions wherever available in the literature.

Numerical investigations have been performed for a couple of symmetric cross-ply laminated, sandwich, simply supported and clamped beams subjected to the transverse sinusoidal loads on the top face. The boundary conditions have been tabulated in Table 1 and material properties used are presented in Table 2. Only half of the beam has been considered in analysis due to symmetry for the examples discussed in this paper. Details of mesh used in various numerical examples have been presented in Table 3.

Following normalization factors have been used in the present work for a proper comparison of results:

\[ \tilde{x}_s = \frac{\sigma_s (a/2, 0)}{q_0}, \quad \tilde{a}_s = \frac{\sigma_s (a/2, Z)}{q_0}, \quad \tilde{v}_{xz} = \frac{\tau_{xz} (0, Z)}{q_0}, \quad (21) \]

Further, in all the numerical examples, beams have been subjected to transverse sinusoidal load of intensity

\[ \hat{q}(X) = q_0 \frac{\pi X}{a}, \quad (22) \]

where \(q_0\) represents the maximum amplitude of distributed load.

### Table 2
Material properties of graphite/epoxy composite

<table>
<thead>
<tr>
<th>Material</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphite epoxy lamina</td>
<td>(E_1 = 172.4 \text{ GPa (25 x 10}^6\text{ psi)})</td>
</tr>
<tr>
<td></td>
<td>(E_2 = 6.89 \text{ GPa (10}^6\text{ psi)})</td>
</tr>
<tr>
<td></td>
<td>(G_{12} = G_{13} = 3.45 \text{ GPa (0.5 x 10}^6\text{ psi)})</td>
</tr>
<tr>
<td></td>
<td>(G_{23} = 1.378 \text{ GPa (0.2 x 10}^6\text{ psi)})</td>
</tr>
<tr>
<td></td>
<td>(v_{12} = v_{13} = v_{23} = 0.25)</td>
</tr>
<tr>
<td>Core</td>
<td>(E_1 = 3.45 \text{ GPa (5 x 10}^4\text{ psi)})</td>
</tr>
<tr>
<td></td>
<td>(E_2 = E_3 = 0.276 \text{ GPa (4 x 10}^4\text{ psi)})</td>
</tr>
<tr>
<td></td>
<td>(G_{12} = G_{13} = 0.414 \text{ GPa (6 x 10}^4\text{ psi)})</td>
</tr>
<tr>
<td></td>
<td>(G_{23} = 0.1104 \text{ GPa (1.6 x 10}^4\text{ psi)})</td>
</tr>
<tr>
<td></td>
<td>(v_{12} = v_{13} = v_{23} = 0.25)</td>
</tr>
</tbody>
</table>

### Table 1
Boundary conditions

<table>
<thead>
<tr>
<th>Edge</th>
<th>BC on displacement field</th>
<th>BC on stress field</th>
</tr>
</thead>
<tbody>
<tr>
<td>At (X = 0)</td>
<td>(u = 0)</td>
<td>(\tau_{xz} = 0)</td>
</tr>
<tr>
<td>At (X = a/2)</td>
<td>(u = 0)</td>
<td>(\sigma_z = q(X), \tau_{xz} = 0)</td>
</tr>
<tr>
<td>At (Z = t/2)</td>
<td>(-)</td>
<td>(\sigma_z = \tau_{xz} = 0)</td>
</tr>
<tr>
<td>At (Z = -t/2)</td>
<td>(-)</td>
<td>()</td>
</tr>
</tbody>
</table>

Boundary conditions for cross-ply laminated beam (fixed support condition) (state of plane-strain)

<table>
<thead>
<tr>
<th>Edge</th>
<th>BC on displacement field</th>
<th>BC on stress field</th>
</tr>
</thead>
<tbody>
<tr>
<td>At (X = 0)</td>
<td>(u = w = 0)</td>
<td>(\sigma_z = 0)</td>
</tr>
<tr>
<td>At (X = a/2)</td>
<td>(u = 0)</td>
<td>(\tau_{xz} = 0)</td>
</tr>
<tr>
<td>At (Z = t/2)</td>
<td>(-)</td>
<td>(\sigma_z = q(X), \tau_{xz} = 0)</td>
</tr>
<tr>
<td>At (Z = -t/2)</td>
<td>(-)</td>
<td>(\sigma_z = \tau_{xz} = 0)</td>
</tr>
</tbody>
</table>

Note: ‘–’ indicates no boundary condition imposed on that DOF.
The elasticity solution for composite laminates under cylindrical bending by Pagano [14], FE solution using multilayer plate bending element by Engblom and Ochoa [41] and Liou and Sun [42], higher order FE solutions by Lo et al. [13] and closed-form solution for bending of laminated plates by Ren [43] have been considered for comparison of results obtained from the present FE model. FE solution using HZZT (EMZC3) and higher order multilayer plate bending (LM4) based on Reissner’s mixed variational theorem by Carrera and Demasi [20,21] have also been used for comparison. On the basis of extensive numerical experimentation with different lay-ups, a minimum of $S^{1.4} + 1$ number of elements along X-direction and element aspect ratio $L_x/L_z$ between 10 and 15 have been found to provide a converging solution.

**Example 1.** A unidirectionally fiber reinforced simply supported beam has been considered. The normalized in-plane stress, $\bar{\sigma}_x(a/2, z/h)$, the maximum transverse stress, $\bar{\tau}_{xz}(\text{max})$ and the transverse displacement, $\bar{w}(a/2, 0)$ have been presented in Table 4 for aspect ratios $S = 4$ and 10. Variation of the transverse displacement ($\bar{w}$) with different aspect ratios has been presented in Fig. 3(a), whereas variation of the normalized in-plane normal stress ($\bar{\sigma}_x$) and the transverse shear stress ($\bar{\tau}_{xz}$) through the thickness for aspect ratio $S = 4$ have been presented in Figs. 3(b) and (c), respectively. Results obtained using present formulation match very well with the elasticity solutions presented by Pagano [14], thereby validating the methodology adopted for the formulation of the present hybrid-interface FE model.

**Example 2.** Bending of a simply supported bidirectional (coupled) ($0^\circ/90^\circ$) laminated beam has been considered in this example. Both the laminae have identical thickness. The normalized in-plane stress, $\bar{\sigma}_x(a/2, \pm h/2)$, the maximum transverse stress, $\bar{\tau}_{xz}(\text{max})$ and the transverse displacement $\bar{w}(a/2, 0)$ have been compared in Table 5 for aspect ratios $S = 4$ and 10. Variation of the normalized transverse displacement ($\bar{w}$) against the aspect ratio has been presented in Fig. 4(a). On the other hand, variation of the in-plane normal stress ($\bar{\sigma}_x$), the transverse normal ($\bar{\tau}_{xz}$), the shear stress ($\bar{\tau}_{xy}$) and the in-plane displacement ($\bar{u}$) through the thickness of the beam with $S = 4$ have been shown in Figs. 4(b–e). Almost all the results have been found to match very well with the elasticity solutions given by Pagano [14]. This shows effectiveness of the present formulation in handling coupled/unsymmetric beams. Further, the example highlights capability of the formulation to estimate accurate values of the transverse stresses ensuring the continuity at interface.

**Example 3.** A simply supported symmetric three-ply ($0^\circ/90^\circ/0^\circ$) laminated beam with equal thickness of each layer has been analyzed in this example. The normalized in-plane stress, $\bar{\sigma}_x(a/2, \pm h/2)$, the maximum transverse stress, $\bar{\tau}_{xz}(\text{max})$ and the transverse displacement $\bar{w}(a/2, 0)$ have been

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### Table 3
Mesh used in various numerical examples

<table>
<thead>
<tr>
<th>Problem description</th>
<th>Discretization using no. of elements in X-dir</th>
<th>Z-dir</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simply supported orthotropic beam $S = 4$ (with symmetry)</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>Simply supported orthotropic beam $S = 10$ (with symmetry)</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Simply supported unsymmetric $0^\circ/90^\circ$ $S = 4$ (with symmetry)</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>Simply supported unsymmetric $0^\circ/90^\circ$ $S = 10$ (with symmetry)</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Simply supported cross-ply beam $0^\circ/90^\circ/0^\circ$ $S = 4$ (with symmetry)</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>Simply supported cross-ply beam $0^\circ/90^\circ/0^\circ$ $S = 10$ (with symmetry)</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Simply supported symmetric five layered beam ($0^\circ/90^\circ$/Core/$90^\circ/0^\circ$) $S = 4$ (with symmetry)</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>Simply supported symmetric five layered beam ($0^\circ/90^\circ$/Core/$90^\circ/0^\circ$) $S = 10$ (with symmetry)</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Clamped cross-ply beam $0^\circ/90^\circ/0^\circ$ $S = 4$ (with symmetry)</td>
<td>4</td>
<td>30</td>
</tr>
<tr>
<td>Clamped cross-ply beam $0^\circ/90^\circ/0^\circ$ $S = 10$ (with symmetry)</td>
<td>10</td>
<td>30</td>
</tr>
</tbody>
</table>

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### Table 4
Comparison of the normalized stresses and the displacement for a simply supported single lamina orthotropic beam subjected to sinusoidal loads

<table>
<thead>
<tr>
<th>Aspect ratio $S$</th>
<th>Stress or displacement</th>
<th>Present analysis</th>
<th>Pagano [14]</th>
<th>Ren [43]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$\bar{\sigma}_x(a/2, h/2)$</td>
<td>14.5623</td>
<td>14.4091</td>
<td>14.0176</td>
</tr>
<tr>
<td></td>
<td>$\bar{\sigma}_x(a/2, -h/2)$</td>
<td>-13.7437</td>
<td>-13.5693</td>
<td>-13.152</td>
</tr>
<tr>
<td></td>
<td>$\bar{\tau}_{xz}(\text{max})$</td>
<td>1.7638</td>
<td>1.7324</td>
<td>1.7456</td>
</tr>
<tr>
<td></td>
<td>$\bar{W}(a/2, 0)$</td>
<td>1.9496</td>
<td>1.9490</td>
<td>1.8837</td>
</tr>
<tr>
<td>10</td>
<td>$\bar{\sigma}_x(a/2, h/2)$</td>
<td>65.8314</td>
<td>65.6920</td>
<td>65.1600</td>
</tr>
<tr>
<td></td>
<td>$\bar{\sigma}_x(a/2, -h/2)$</td>
<td>-65.706</td>
<td>-65.5130</td>
<td>-64.9800</td>
</tr>
<tr>
<td></td>
<td>$\bar{\tau}_{xz}(\text{max})$</td>
<td>4.7360</td>
<td>4.6828</td>
<td>4.3930</td>
</tr>
<tr>
<td></td>
<td>$\bar{W}(a/2, 0)$</td>
<td>0.7320</td>
<td>0.7319</td>
<td>0.7216</td>
</tr>
</tbody>
</table>
present theory are in very good agreement with the

Note: ‘−’ result not available.

Table 5
Comparison of the normalized stresses and the displacement for a simply supported two-ply (0°/90°) coupled laminated beam

<table>
<thead>
<tr>
<th>Aspect ratio $S$</th>
<th>Stress/displacement</th>
<th>Present analysis</th>
<th>Pagano [14]</th>
<th>Liou and Sun [42]</th>
<th>Engblom and Ochoa [41]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$\bar{\sigma}_x(a/2, h/2)$</td>
<td>3.8786</td>
<td>3.8359</td>
<td>–</td>
<td>3.01</td>
</tr>
<tr>
<td></td>
<td>$\bar{\sigma}_x(a/2, -h/2)$</td>
<td>-30.4281</td>
<td>-30.0293</td>
<td>–</td>
<td>-27.4</td>
</tr>
<tr>
<td></td>
<td>$\bar{\tau}_{xz}(\text{max})$</td>
<td>2.8204</td>
<td>2.7219</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$\bar{W}(a/2, 0)$</td>
<td>4.6916</td>
<td>4.6954</td>
<td>4.703</td>
<td>5.63</td>
</tr>
<tr>
<td>10</td>
<td>$\bar{\sigma}_x(a/2, h/2)$</td>
<td>19.8835</td>
<td>19.8292</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$\bar{\sigma}_x(a/2, -h/2)$</td>
<td>-177.2260</td>
<td>-176.5280</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$\bar{\tau}_{xz}(\text{max})$</td>
<td>7.2603</td>
<td>7.2679</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>$\bar{W}(a/2, 0)$</td>
<td>2.9554</td>
<td>2.9538</td>
<td>2.954</td>
<td>3.335</td>
</tr>
</tbody>
</table>

Note: ‘−’ result not available.

Table 6 for aspect ratios $S = 4$ and 10. Variation of the in-plane normal stress ($\bar{\sigma}_x$), the transverse normal ($\bar{\sigma}_z$), the shear stress ($\bar{\tau}_{xz}$) and the in-plane displacement field ($\bar{u}$) through the thickness of the beam with $S = 4$ has been presented in Figs. 5(a–d). It can be seen that the results from the present theory are in very good agreement with the
Fig. 4. Variation of the normalized (a) transverse displacement $\bar{w}(a/2, 0)$ with aspect ratio $S$; (b) in-plane normal stress ($\bar{\sigma}_x$); (c) transverse normal stress ($\bar{\sigma}_z$); (d) transverse shear stress ($\bar{\tau}_{xz}$); and (e) in-plane displacement ($\bar{u}$) through the thickness for a simply supported two-ply (0°/90°) laminated beam subjected to sinusoidal loading for $S = 4$. 
Table 6
Comparison of the normalized stresses and the displacement for a simply supported cross-ply \((0^\circ/90^\circ/0^\circ)\) laminated beam subject to sinusoidal loads

<table>
<thead>
<tr>
<th>Aspect ratio (S)</th>
<th>Stress/displacement</th>
<th>Present analysis</th>
<th>Pagano [14]</th>
<th>Liou and Sun [42]</th>
<th>Engblom and Ochoa [41]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(\tilde{\sigma}_x (a/2, h/2))</td>
<td>19.0055</td>
<td>18.8080</td>
<td>16.2</td>
<td>10.12</td>
</tr>
<tr>
<td></td>
<td>(\tilde{\sigma}_y (a/2, -h/2))</td>
<td>-18.3065</td>
<td>-18.1044</td>
<td>-15.6</td>
<td>-10.33</td>
</tr>
<tr>
<td></td>
<td>(\tilde{\tau}_{xz} (\text{max}))</td>
<td>1.6313</td>
<td>1.5827</td>
<td>1.68</td>
<td>1.76</td>
</tr>
<tr>
<td></td>
<td>(\tilde{W} (a/2, 0))</td>
<td>2.8863</td>
<td>2.8872</td>
<td>2.899</td>
<td>3.52</td>
</tr>
<tr>
<td>10</td>
<td>(\tilde{\sigma}_x (a/2, h/2))</td>
<td>73.8129</td>
<td>73.6704</td>
<td>–</td>
<td>63.78</td>
</tr>
<tr>
<td></td>
<td>(\tilde{\sigma}_y (a/2, -h/2))</td>
<td>-73.7973</td>
<td>-73.6304</td>
<td>–</td>
<td>-63.67</td>
</tr>
<tr>
<td></td>
<td>(\tilde{\tau}_{xz} (\text{max}))</td>
<td>4.2350</td>
<td>4.2393</td>
<td>–</td>
<td>4.21</td>
</tr>
<tr>
<td></td>
<td>(\tilde{W} (a/2, 0))</td>
<td>0.9321</td>
<td>0.93164</td>
<td>0.9315</td>
<td>1.24</td>
</tr>
</tbody>
</table>

Note: ‘–’ result not available.

Fig. 5. Variation of the normalized (a) in-plane normal stress \(\tilde{\sigma}_x\); (b) transverse shear stress \(\tilde{\tau}_{xz}\); (c) transverse normal stress \(\tilde{\sigma}_x\); and (d) in-plane displacement \(\tilde{u}\) through the thickness of a simply supported symmetric cross-ply \((0^\circ/90^\circ/0^\circ)\) laminated beam subjected to sinusoidal loading for \(S = 4\).
elasticity solution [14]. It should also be noted that the hybrid stress models of Liou and Sun [42] and Carrera and Damasi [20,21] have yielded good results (see Figs. 5(a–d)). However, the displacement-based models (viz. [13,41]) could not predict the in-plane and the transverse stress (see Figs. 5(a–c)) as well as variation of the in-plane displacement (see Fig. 5(d)). This highlights the superiority of the present hybrid model over displacement-based models in analysis of laminated structures.

Example 4. Bending of a simply supported symmetric five layered (0°/90°/Core/90°/0°) laminated beam with equal thickness of each layer has been considered in this example. It has been observed that minimum four element rows for each lamina in thickness direction are required for converging solution. The normalized in-plane stress $\bar{\sigma}_x(a/2, h/2)$, the maximum transverse stress, $\bar{\tau}_{xz}(\text{max})$ and the transverse displacement, $\bar{w}(a/2, 0)$ have been compared in Table 7 for aspect ratios $S = 4$ and 10. Variations of the in-plane normal stress ($\bar{\sigma}_x$), the transverse normal and the shear stresses ($\bar{\tau}_z$ and $\bar{\tau}_{xz}$) as well as the in-plane displacement ($\bar{u}$) through the thickness of the beam with $S=4$ have been shown in Figs. 6(a–d). Results are in excellent agreement with the elasticity solution [14].

<table>
<thead>
<tr>
<th>Aspect ratio $S$</th>
<th>Stress/displacement</th>
<th>Present analysis</th>
<th>Pagano [14]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$\bar{\sigma}_x(a/2, h/2)$</td>
<td>27.3628</td>
<td>27.0806</td>
</tr>
<tr>
<td></td>
<td>$\bar{\sigma}_x(a/2, -h/2)$</td>
<td>-26.7900</td>
<td>-26.5095</td>
</tr>
<tr>
<td></td>
<td>$\bar{\tau}_{xz}(\text{max})$</td>
<td>1.6041</td>
<td>1.53693</td>
</tr>
<tr>
<td></td>
<td>$\bar{w}(a/2, 0)$</td>
<td>5.34779</td>
<td>5.34856</td>
</tr>
<tr>
<td>10</td>
<td>$\bar{\sigma}_x(a/2, h/2)$</td>
<td>94.1706</td>
<td>93.9936</td>
</tr>
<tr>
<td></td>
<td>$\bar{\sigma}_x(a/2, -h/2)$</td>
<td>-94.2788</td>
<td>-94.0957</td>
</tr>
<tr>
<td></td>
<td>$\bar{\tau}_{xz}(\text{max})$</td>
<td>3.8408</td>
<td>3.66198</td>
</tr>
<tr>
<td></td>
<td>$\bar{w}(a/2, 0)$</td>
<td>1.4632</td>
<td>1.46336</td>
</tr>
</tbody>
</table>

Fig. 6. Variation of the normalized (a) in-plane normal stress ($\bar{\sigma}_x$); (b) transverse shear stress ($\bar{\tau}_{xz}$); (c) transverse normal stress ($\bar{\tau}_z$); and (d) in-plane displacement ($\bar{u}$) through the thickness of a simply supported (0°/90°/Core/90°/0°) laminated beam subjected to sinusoidal loading for $S = 4$. 
Table 8
Normalized stresses and displacement for a clamped cross-ply (0°/90°/0°) laminated beam subjected to sinusoidal load

<table>
<thead>
<tr>
<th>Stress/displacement</th>
<th>Present analysis (S = 4)</th>
<th>Present analysis (S = 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{\sigma}_1 (a/2, h/2) )</td>
<td>10.9630</td>
<td>33.9520</td>
</tr>
<tr>
<td>( \bar{\sigma}_2 (a/2, -h/2) )</td>
<td>-10.1027</td>
<td>-33.6403</td>
</tr>
<tr>
<td>( \tau_{xz} ) (max)</td>
<td>1.6400</td>
<td>4.0876</td>
</tr>
<tr>
<td>( \bar{W} (a/2, 0) )</td>
<td>2.88692</td>
<td>0.93157</td>
</tr>
</tbody>
</table>

Fig. 7. (a) Variation of the normalized transverse displacement (\( \bar{w} \)) with aspect ratio \( S \) at (a/2, 0); (b–d) through thickness variation of the normalized (b) transverse shear stress (\( \tau_{xz} \)); (c) transverse normal stress (\( \bar{\sigma}_2 \)); and (d) in-plane normal stress (\( \bar{\sigma}_1 \)) of cross-ply (0°/90°/0°) clamped laminated beam subjected to sinusoidal loads for \( S = 4 \) and 10.

Example 5. The beam considered in Example 3 is analyzed for clamped support condition instead of simple supports to provide new benchmark solutions and also to show the capability of the present formulation to handle problems with high stress gradient. The normalized in-plane stress, \( \bar{\sigma}_1 (a/2, \pm h/2) \), the

The example shows the capability of present formulation to handle sandwich beams effectively.
maximum transverse stress, $\tau_{xz}(\text{max})$ and the transverse displacement, $\bar{w}(a/2, 0)$ have been presented in Table 8 for aspect ratios $S = 4$ and 10. Variation of the transverse displacement ($\bar{w}$) with different aspect ratios has been shown in Fig. 7(a). Variation of the transverse shear and the normal stresses ($\tau_{xz}$ and $\sigma_z$), as well as the in-plane normal stress ($\sigma_x$) through the thickness of the beams (with $S = 4$ and 10) have been presented in Figs. 7(b–d). The variation of the transverse stresses ($\tau_{xz}$) shows the presence of extremely high stress gradient at the top and bottom surfaces and at layer interfaces. A very refined theoretical model can only handle the presence of such a high stress gradient. The present formulation has shown its capability of handling such situations.

4. Conclusions

A novel hybrid-interface FE formulation has been presented based on the minimum energy principle for analysis of thick/thin laminated composite beams. The hybrid-interface elements are used at lamina interfaces in combination with conventional displacement-based elements ensuring their compatibility. Fundamental elasticity relationship between stress, strain and displacement fields within the elastic continuum have been ensured in the formulation. The transverse stress components have been considered as primary variables at interfaces thus ensuring it is through thickness continuity in addition to continuity of displacement components. Hence the present formulation can be implemented for realistic estimate of First Ply Failure load as well as initiation of delamination. Present formulation also enables the specification of homogeneous/non-homogeneous stress boundary conditions at beam faces and in delaminated region. This feature can be used for a reliable estimate of design/failure loads. Excellent agreement of results with the elasticity solution suggest that the formulation is capable of dealing with laminated composite and sandwich beams for a variety of support and loading conditions. A very refined formulation like the present one is required to handle problems with high stress gradient of the transverse stresses.

Acknowledgment

Constructive comments of the reviewers are gratefully acknowledged.

References


