Template generation for continuous transfer functions using interval analysis[☆]

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Abstract

This paper presents a template generation algorithm for transfer functions that are continuous in the uncertain system parameters. The algorithm is developed using interval mathematics. The generated templates are of arbitrary accuracy, safe, and reliable. Further, if the magnitude and phase are continuously differentiable functions of the parameters, then the algorithm can be speeded up considerably with the help of an additional step based on the interval Gauss–Seidel method. The convergence, finite termination, safety, and reliability properties of the algorithm are proven, and an example is provided to demonstrate its versatility.

Keywords: Frequency responses; Image sets; QFT; Templates; Transfer functions; Uncertain linear systems; Value sets

1. Introduction

A plant *template* is a set of complex numbers representing the frequency response at a fixed frequency of an uncertain plant transfer function. Plant templates are used in several approaches to analysis and design of robust control systems, for example, Bhattacharyya, Chapellat and Keel (1995) and QFT of Horowitz (1991). Templates are also referred to as *value sets* or *image sets* in the literature, and the problem of computing the template at a given frequency is known as the template generation problem or the value set computation problem.

Several approaches to solve the template generation problem exist. However, these existing template generation methods (with the exception of grid-based ones) are restricted in scope to particular kinds of correlations and dependencies in the system parameters: independent parameters (Bailey, Panzer & Gu, 1988; Bailey & Panzer, 1988; Fialho, Pande & Nataraj, 1992; Hwang & Chen,

1995; Lan, 1994); affine parametric dependency (Bailey & Hui, 1989; Bartlett, 1990; Fu, 1990); multi-linear dependency (Bhattacharyya et al., 1995); and special non-linear dependency (Barmish & Tempo, 1995). Further, in these methods there are inherent restrictions to transfer functions of particular forms, such as rational, tree-structured decomposable (Barmish, Ackermann & Hu, 1990; Ackermann & Sienel, 1991; Eszter & Pena, 1994); factored real form (Gutman, Baril & Neumann, 1994), and other forms (Fu, Dasgupta & Blondel, 1995; Isogai, Ohta & Haneda, 1993; Ohta, 1996; Ohta, Gong & Haneda, 1994). A few of these algorithms (Lasky & Ravani, 1997) carry out convex hull approximations of the template, which could be quite conservative in cases where the actual template is non-convex. Grid-based methods (Cohen, Nordin & Gutman, 1995; Horowitz, 1991), are applicable with no known restrictions on the kind of parametric dependency or form of transfer function. Unfortunately, a limitation with any grid-based method is that it is potentially risky, in the sense that some critical points of the templates could be missed due to the very nature of the grid process.

This paper presents a template generation algorithm based on interval analysis methods. The merits of the proposed algorithm are as follows. (i) The algorithm has a vast scope — it is applicable to any transfer function whose magnitude and phase are *continuous*

functions of the uncertain system parameters. These continuous functions may be computable as explicit formulas, recursively, via numerical algorithms, or computer codes. Further, these functions could involve any combination and compositions of the elementary arithmetic operations and unary functions (including all pre-declared functions of a programming language, such as $\sin$, $\cos$, $\exp$, $\log$, $\sqrt{\cdot}$, power , and $\arctan$). (ii) The algorithm can generate, in theory, templates of arbitrary accuracy (in practice, the accuracy is limited by the available machine precision and memory storage). (iii) The algorithm, when implemented on an interval arithmetic compiler, generates a so-called *safe* and *reliable* template that includes *all* the actual template points at the same frequency, despite all kinds of computational errors, such as round-off and truncation. This key aspect guarantees that the resulting controller design is indeed valid over the *entire* plant set. (iv) If the magnitude and phase functions are continuously differentiable, then, this property can be exploited to considerably speed up the algorithm, and to obtain a significantly smaller set of solution rectangles. (v) No rational function approximation of the non-rational transfer function is performed in the algorithm.

This paper is organized as follows. Section 2 gives the essentials of interval mathematics relevant to the work. Section 3 presents the template generation algorithm. Section 4 gives theoretic properties of the algorithm. An illustrative example follows in Section 5. In Section 6, the conclusions of the work are given.

2. Real interval mathematics

This section collects some preliminaries of interval analysis. A real interval is a closed and bounded set of real numbers $X := [\underline{X}, \bar{X}] := \{x \in \mathbb{R} : \underline{X} \leq x \leq \bar{X}\}$ where, $\underline{X}$ and $\bar{X}$ are the lower and upper endpoints of the interval X . The set of all real intervals is denoted by $I\mathbb{R}$. Two intervals are equal if their corresponding endpoints are equal. The intersection of two intervals X and Y is empty, $X \cap Y = \emptyset$, if either $\underline{X} > \bar{Y}$ or $\underline{Y} > \bar{X}$. Else, the intersection of X and Y is again an interval $X \cap Y := [\max(\underline{X}, \underline{Y}), \min(\bar{X}, \bar{Y})]$. The set inclusion: $X \subseteq Y$ if and only if $\underline{Y} \leq \underline{X}$ and $\bar{X} \leq \bar{Y}$. Further, the width of an interval X is defined as $w(X) := \bar{X} - \underline{X}$, and the mean as $m(X) := (\underline{X} + \bar{X})/2$.

An interval vector is a vector whose elements are intervals. The set of all n -dimensional real interval vectors is denoted by $I\mathbb{R}^n$. The notation $X := (X_1, \dots, X_n)$, $X \in I\mathbb{R}^n$ is used. The relations $\in$, $=$, and $\subseteq$, union, intersection, and mean are all defined component wise. ‘Box’ is a synonym for an interval vector, and ‘rectangle’ for a 2-dim interval vector.

In addition to the elementary arithmetic operations, there are further common, mostly unary, operations on intervals (Moore, 1979):

Definition 2.1. Let $r(x)$ be a continuous unary operation defined on $\mathbb{R}$. Then, a unary operation on $I\mathbb{R}$ is defined as $r(X) := [\min_{x \in X} \{r(x)\}, \max_{x \in X} \{r(x)\}]$.

2.1. Interval extension and range enclosures

Definition 2.2. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function defined over a box $X \in I\mathbb{R}^n$. Then, the range of f over X is denoted as $\bar{f}(X)$. That is, $\bar{f}(X) := \{f(x) : x \in X\}$.

Definition 2.3 (Interval extension). A function $F: I\mathbb{R}^n \rightarrow I\mathbb{R}^m$ is said to be an interval extension of function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ provided $\bar{f}(X) \subseteq F(X)$ for all boxes $X \in I\mathbb{R}^n$ within the domain of F .

Definition 2.4 (Natural interval extension). If $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function computable as an expression, algorithm, or computer program involving the four elementary arithmetic operations interspersed with evaluations of unary functions, then a *natural interval extension* of f , whose value over a box X is denoted by $F(X)$, is obtained by replacing each occurrence of each component x_i of x by the corresponding interval component X_i of X , by executing all operations according to standard interval arithmetic rules, and by computing the exact ranges of the unary functions as in Definition 2.1.

Definition 2.5. An interval extension F , defined over a domain X , is said to be inclusion monotonic if $Y \subseteq X \Rightarrow F(Y) \subseteq F(X)$.

Two key properties of interval extensions are

Theorem 2.1 ((Moore, 1979) (Inclusion monotonicity)). *Natural interval extensions are inclusion monotonic.*

Theorem 2.2 ((Moore, 1979) (Inclusion property)). *Let $F(X)$ denote a natural interval extension of $f(x)$. Then, $\bar{f}(X) \subseteq F(X)$.*

2.2. Interval topology

Definition 2.6. Let A, B be compact, non-empty subsets of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Define

$$d_0(x, B) := \min_{b \in B} \|x - b\|_2, \quad d_0(A, B) := \max_{a \in A} d_0(a, B),$$

$$d(A, B) := \max\{d_0(A, B), d_0(B, A)\}.$$

The Hausdorff-distance d is a metric for the sets of compact non-empty subsets of $\mathbb{R}^n$. With this metric, a topology is defined, and convergence, etc., can be defined in the usual manner. In particular, the following definition can be made:

Definition 2.7. A sequence $(X_k)_{k=1}^\infty$ of interval vectors $X_k \in I\mathbb{R}^n$ is said to converge to a point $z \in \mathbb{R}^n$ if $d(X_k, z) \rightarrow 0$ as $k \rightarrow \infty$.

2.3. Interval linear systems

Consider the rectangular interval system $AX = B$, where $A \in \mathcal{IR}^{m \times n}$ and $B \in \mathcal{IR}^m$, $m \leq n$. The solution set of this system is that set $\sum(A, B) \subseteq \mathcal{R}^n$ such that if $x \in \sum(A, B)$, there exists an $\tilde{A} \in A$ and a $\tilde{B} \in B$ such that $\tilde{A}x = \tilde{B}$. The so-called Generalized Gauss–Seidel method (GGs) has been employed in Neumaier (1988) to find such a solution set (preconditioning of the system is strongly suggested). The reader is referred to Neumaier (1988) for details of the GGs method. The following adaptation of the result in Neumaier (1990) is required in the sequel.

Theorem 2.3. Consider the interval linear system $AX = B$ where $A \in \mathcal{IR}^{m \times n}$ and $B \in \mathcal{IR}^m$, $m \leq n$. Let $\sum(A, B)$ denote the solution set of this system. Then

$$\sum(A, B) \cap X^C \subseteq \sum(CA, CB) \cap X^C \subseteq X_{\text{GGs}}$$

where, the preconditioning matrix $C \in \mathcal{R}^{m \times m}$ is an approximate inverse of a matrix composed of m linearly independent columns of some $\tilde{A} \in A$, X_{GGs} is the solution obtained using GGs for the preconditioned interval system, and X^C is the initial interval vector (box) in which the solution is sought.

3. Algorithm for template generation

Consider a system represented by the transfer function $g(s, \lambda)$, where $\lambda = \{\lambda_1, \dots, \lambda_l\}$ is a real vector of the fundamental system parameters, and s is the Laplace variable. Suppose the parameters λ_i vary independently over given real intervals Λ_i^0 , so that $\Lambda^0 := \{\Lambda_1^0, \dots, \Lambda_l^0\}$ is a box.

Definition 3.1 (In above notation). The angle and magnitude functions for the transfer function $g(s, \lambda)$ are defined as

$$h_{\text{ang}}(\lambda) := \arg\{g(s = j\omega, \lambda)\}, \quad h_{\text{mag}}(\lambda) := |g(s = j\omega, \lambda)| \quad (1)$$

where, $j = \sqrt{-1}$, and ω is the frequency variable (whose value is fixed). Further, define $h(\lambda) := (h_{\text{ang}}(\lambda), h_{\text{mag}}(\lambda))$. Then, the set $\mathcal{G} := \bar{h}(\Lambda^0) := \{h(\lambda), \lambda \in \Lambda^0\}$ is called as the template of $g(s, \lambda)$ at the given ω .

Definition 3.2. Let Λ denote the interval vector variable corresponding to the real vector variable λ . Then, $H_{\text{ang}}(\Lambda)$ will denote a natural interval extension of h_{ang} over Λ . Similarly for $H_{\text{mag}}(\Lambda)$.

Definition 3.3. Let $n = l + 2$, where l is the length of the λ vector. Define the box $D \in \mathcal{IR}^n$ as follows:

$$D := (H_{\text{ang}}(\Lambda^0), H_{\text{mag}}(\Lambda^0), \Lambda^0). \quad (2)$$

Definition 3.4. Let $x = (x_1, x_2, \lambda) \in D$, and define the function

$$f(x) := (x_1 - h_{\text{ang}}(\lambda), x_2 - h_{\text{mag}}(\lambda)). \quad (3)$$

Further, let X denote the interval vector variable corresponding to x . Then, $F(X)$ will denote a natural interval extension of f over X .

Definition 3.5. Let $y := (x_1, x_2)$, so that $x = (y, \lambda)$. In terms of corresponding interval variables, let $Y := (X_1, X_2)$ so that $X = (Y, \Lambda)$. Then, Y will be called as the ‘constituent rectangle’ of the subbox X .

Definition 3.6. Suppose f is differentiable on D . Then, the Jacobi matrix (arranged as a $2 \times n$ matrix) of $f(x)$ in (3) will be denoted by $f'(x)$; further, $F'(X)$ will denote any interval Jacobi matrix whose components are natural interval extensions over X of corresponding components of f' .

Throughout this work, the following will be assumed.

Assumption 1. f is continuous, with its natural interval extension F well-defined, on D .

The template generation algorithm is presented next.

Algorithm (Template Generation Algorithm).

Inputs: An expression for the transfer function $g(s, \lambda)$, Λ^0 , ω , and a parameter ε_x specifying the maximum width of each solution constituent rectangle.

Output: The approximating template $\mathcal{G}^{\text{alg}}$ constructed in step 3a below.

Note: The algorithm is to be executed in the order given below, except when otherwise indicated.

BEGIN Algorithm

1. (Symbolic manipulation phase)
 - (a) From the transfer function expression $g(s, \lambda)$, obtain the angle and magnitude functions $h_{\text{ang}}(\lambda)$, $h_{\text{mag}}(\lambda)$ in (1).
 - (b) Find the natural interval extensions $H_{\text{ang}}(\Lambda)$, $H_{\text{mag}}(\Lambda)$ from the expressions for $h_{\text{ang}}(\lambda)$, $h_{\text{mag}}(\lambda)$, respectively.
 - (c) Form the function $f(x)$ in (3), and find the natural interval extension $F(X)$ from the expression for $f(x)$.
2. (Initialization phase)
 - (a) Compute D in (2).
 - (b) Set: $X^0 \leftarrow D$.
 - (c) Set: $k \leftarrow 0$.
 - (d) Initialize lists: $\mathcal{L}^{\text{sol}} \leftarrow \{\}$, $\mathcal{L} \leftarrow \{X^0\}$.
3. (Start a new iteration)
 - (a) If $\mathcal{L}$ is empty, construct $\mathcal{G}^{\text{alg}} := \bigcup_{Y_i \in \mathcal{L}^{\text{sol}}} Y_i$, and EXIT.
 - (b) $k \leftarrow k + 1$

- (c) Choose the first box from $\mathcal{L}$, call it X , and delete X from $\mathcal{L}$.
4. (Test phase and solution list)
- (a) (Exclusion test) Evaluate $F(X)$. If $0 \notin F(X)$, then discard X and return to Step 3a.
- (b) (GGs Step. Executed only if f is continuously differentiable, and natural interval extension F' is well-defined, on D) Using automatic differentiation (Moore, 1979) evaluate the interval Jacobi matrix $F'(X)$. Next, using GGS, find the solution set X_{GGS} of the homogenous interval linear equation $AX = 0$, where
- $$A = (F'(X); f^T(\tilde{z})), \quad \tilde{X} = (X - \tilde{z}; 1)^T, \\ \tilde{z} = m(X)$$
- If GGS returns an empty solution set, then discard X and return to Step 3a. Else, set $X \leftarrow X_{\text{GGS}}^T + \tilde{z}$.
- (c) If $w(X) < \varepsilon_x$, enter Y into $\mathcal{L}^{\text{sol}}$ (note that $X = (Y, \Lambda)$) and return to Step 3a.
5. (Bisection phase)
- (a) Choose a coordinate direction parallel to which X has an edge of maximum width.
- (b) Bisect X along the above chosen coordinate direction to get two sub-boxes X^A, X^B such that $X = X^A \cup X^B$.
- (c) Enter X^A, X^B into $\mathcal{L}$ such that the widths of boxes in $\mathcal{L}$ do not increase.
6. (End current iteration) Return to Step 3a.
- END Algorithm.

Remark 3.1. The generality of the algorithm is emphasized. Assumption 1 means that the algorithm accepts as functional parameters magnitude and phase functions that are continuous and have natural interval extensions well-defined on D . Now, such extensions can be constructed for any magnitude or phase function expressible as combination and compositions of the elementary arithmetic operations and unary functions (Moore, 1979, pg. 3). The unary functions include absolute value, square, square root, exponential function, power function, logarithmic functions, trigonometric and inverse trigonometric functions, and hyperbolic and inverse hyperbolic functions. Other functions may also be included in this list, provided they are continuous on each closed interval on which they are defined. Hence, one can obtain natural interval extensions for virtually any magnitude or phase function that can be represented with a computer program (Kearfott, 1996, pg. 12).

Remark 3.2. The algorithm as given above assumed the availability of explicit functional expressions for $h_{\text{ang}}(\lambda)$, $h_{\text{mag}}(\lambda)$, and $f(x)$. This was done only for presentational convenience and is not necessary. Alternate representations for $h_{\text{ang}}(\lambda)$, $h_{\text{mag}}(\lambda)$, and $f(x)$, such as recursive

ones or computer codes, may be used. The construction of natural interval extensions for these alternate representations is straightforward, and runs completely automatically with the help of interval compilers (Ratschek, 1985); similar remarks apply to the task of automatic differentiation in step 4b.

Remark 3.3. If f is continuously differentiable on D , then this property can be used to speed up the algorithm through execution of Step 4b. A considerably smaller set of solution rectangles is also usually obtained in this case. Further, in this step, it is usually more effective to perform two sweeps of the GGS method (Neumaier, 1988) and use an interval slope matrix instead of an interval Jacobi matrix (Kearfott 1996).

Remark 3.4. For subsequent robustness analysis and controller design purposes, a more wieldy representation of the template (in terms of a reduced number of rectangles) can be obtained by applying steps 7–9 of the algorithm in Sardar and Nataraj (1997) to $\mathcal{G}^{\text{alg}}$.

4. Theoretic properties of algorithm

The below lemma says that the algorithm is initialized with its initial list $\mathcal{L}$ containing a constituent rectangle that encloses the template.

Lemma 4.1. *Let X^0 be as in Step 2d of the algorithm, and Y^0 be the constituent rectangle of X^0 . Then, $\mathcal{G} \subseteq Y^0$.*

Proof. In Step 2d of the algorithm, $X^0 \leftarrow D = (H_{\text{ang}}(\Lambda^0), H_{\text{mag}}(\Lambda^0), \Lambda^0)$, so $Y^0 \leftarrow (H_{\text{ang}}(\Lambda^0), H_{\text{mag}}(\Lambda^0))$. By the inclusion property (Theorem 2.2), $\bar{h}(\Lambda^0) \subseteq (H_{\text{ang}}(\Lambda^0), H_{\text{mag}}(\Lambda^0))$. Hence, $\mathcal{G} := \bar{h}(\Lambda^0) \subseteq Y^0$. $\square$

The below lemma justifies Step 4a in the algorithm.

Lemma 4.2. *Let $X \subseteq X^0$ be a subbox, where X^0 is as in Step 2b of the algorithm. Suppose, $0 \notin F(X)$. Then, X can be discarded in the algorithm.*

Proof. By hypothesis, $0 \notin F(X)$. Then, from Definitions 3.4 and 3.5, $Y \cap H(\Lambda) = \emptyset$. By the inclusion property in Theorem 2.2, $\bar{h}(\Lambda) \subseteq H(\Lambda)$. Thus, $Y \cap \bar{h}(\Lambda) = \emptyset$, i.e., the constituent rectangle Y of X does not contain any template points. Hence, X is irrelevant in the search for template points, and can be discarded. $\square$

The below two lemmas justify (if executed) Step 4b. in the algorithm.

Lemma 4.3 (If Step 4b is executed). *Let $X \subseteq X^0$ be a subbox, and suppose that the GGS method applied to X*

in Step 4b returns an empty solution set, i.e., $X_{\text{GGS}} = \emptyset$. Then, X can be discarded in the algorithm.

Proof. Suppose $x^* \in X$ is such that $f(x^*) = (f_1(x^*), f_2(x^*)) = 0$. Let $\tilde{z} = m(X)$. Then, the mean value theorem of calculus gives

$$\begin{aligned} 0 &= f_1(x^*) = f_1(\tilde{z}) + f'_1(\zeta_1)(x^* - \tilde{z})^T, \quad \zeta_1 \in X, \\ 0 &= f_2(x^*) = f_2(\tilde{z}) + f'_2(\zeta_2)(x^* - \tilde{z})^T, \quad \zeta_2 \in X. \end{aligned}$$

From Theorem 2.2, $f'_1(\zeta_1) \in F'_1(X), f'_2(\zeta_2) \in F'_2(X)$. Thus, $(x^* - \tilde{z})^T$ must lie in the solution set to

$$F'(X)(X - \tilde{z})^T = -f^T(\tilde{z}). \quad (4)$$

By Theorem 2.3, X_{GGS} got using the GGS method encloses the solution set of (4). Hence, any root x^* of f within X must be within $\tilde{z} + X_{\text{GGS}}^T$. From this, it follows that if $X_{\text{GGS}} = \emptyset$, then there are no roots of f within X , implying that no template points lie in the constituent rectangle Y of X . Hence, X can be discarded in the search for template points. $\square$

Lemma 4.4 (If Step 4b is executed). *Let $X \subseteq X^0$ be a sub-box, and suppose that the GGS method applied to X in Step 4b returns a non-empty solution set, i.e., $X_{\text{GGS}} \neq \emptyset$. Then, the constituent rectangle (of the GGS solution) that is produced retains any template points present in the constituent rectangle Y of X .*

Proof. Continuing the proof in the preceding lemma for the other case $X_{\text{GGS}} \neq \emptyset$, it follows that any template point present in the constituent rectangle Y of X must also be present in the constituent rectangle of $\tilde{z} + X_{\text{GGS}}^T$, proving the assertion of this lemma. $\square$

4.1. Convergence

Its assumed in this sub-section that the termination criterion in the algorithm can never be satisfied (i.e. $\varepsilon_x = 0$) so that the convergence properties can be studied. It is also assumed for the same reason that the list sizes are not a limitation. First, some further quantities are defined.

Definition 4.1. Let $\mathcal{L}_k$ denote the list $\mathcal{L}$ present at the start of the k th iteration of the algorithm, and denote the i th box of this list as $X_{ki} = (Y_{ki}, \Lambda_{ki})$. Define the following unions

$$U_k = \bigcup_{Y_{ki} \in \mathcal{L}_k} Y_{ki}, \quad V_k = \bigcup_{\Lambda_{ki} \in \mathcal{L}_k} \Lambda_{ki}, \quad W_k = \bigcup_{X_{ki} \in \mathcal{L}_k} X_{ki}. \quad (5)$$

Lemma 4.5. $\mathcal{G}$ is a compact set in $\mathbb{R}^2$.

Proof. By Assumption 1, h is continuous on D . Clearly, the parameter box Λ^0 is compact. As the image of a con-

tinuous mapping on a compact set is compact, so $\bar{h}(\Lambda^0)$, i.e., $\mathcal{G}$, is compact. $\square$

Similarly, it can be shown that

Lemma 4.6. *The unions U_k, V_k, W_k , are compact.*

Lemma 4.7. *Let U_k be as in (5). Then, $\mathcal{G} \subseteq \bigcap_1^\infty U_k$.*

Proof. It suffices to show that $\mathcal{G} \subseteq U_k$ for any k . The assertion of the lemma then follows. Consider the algorithm with $k = 1$. Firstly, note that, by Lemma 4.1, all the template points are contained in Y^0 , i.e., $\mathcal{G} \subseteq U_1$. By Lemma 4.2, the discarding process in Step 4a of the algorithm does not delete any template point contained in Y^0 . Further, if Step 4b is executed, then, by Lemma 4.4, the GGS method also does not delete any template point contained in Y^0 . Moreover, no template point in Y^0 can be lost in the bisection phase (because both the sub-boxes resulting from bisection are entered into the list). Thus, at the end of the first iteration, all the template points are retained in U_2 , i.e., $\mathcal{G} \subseteq U_2$. By induction on k , we have $\mathcal{G} \subseteq U_k$ for any $k \Rightarrow \mathcal{G} \subseteq \bigcap_1^\infty U_k$. $\square$

Remark 4.1. From the above lemma, it can be seen that the unions U_k, V_k, W_k , are non-empty compact sets, for any k . Further, $\bigcap_1^\infty U_k$ and $\bigcap_1^\infty V_k$ are non-empty sets.

Lemma 4.8. *Let U_k be as in (5). Then, the sequence $(U_k)_{k=1}^\infty$ has the property $U_1 \supseteq U_2 \supseteq U_3 \cdots$.*

Proof. In the first iteration ($k = 1$) of the algorithm, the box X^0 is picked from the list $\mathcal{L}_1$, (some) irrelevant portions of X^0 removed in the GGS step (if Step 4b is executed), and the remaining bisected; the boxes resulting from the bisection process then replace X^0 in $\mathcal{L}_1$, giving the list $\mathcal{L}_2$ (state of the list at $k = 2$). The proof is completed by induction on k . $\square$

Lemma 4.9. *Let w_k denote the maximum width of boxes of the k th list $\mathcal{L}_k$ generated by the algorithm. Then, $w_k \rightarrow 0$ as $k \rightarrow \infty$*

Proof. Follows from the lemma in Ratschek (1985). $\square$

Lemma 4.10. *Let X and $F(X)$ be as in Definition 3.4. Then, $w(F(X)) \rightarrow 0$ as $w(X) \rightarrow 0$*

Proof. By Theorem 1.5 in Kearfott (1996), all natural interval extensions are inclusion functions of first-order, that is, there is a constant $\alpha > 0$ independent of the box X such that

$$w(F(X)) - w(\bar{f}(X)) \leq \alpha w(X) \quad (6)$$

for all boxes $X \subseteq D$ with $w(X)$ sufficiently small. Since, by Assumption 1, f is continuous on D , $w(\bar{f}(X)) \rightarrow 0$ as

$w(X) \rightarrow 0$. Together with (6), this leads to the assertion of the lemma. $\square$

Definition 4.2. Let $f(x)$ be as in (3), and X^0 be as in Step 2b of the algorithm. Define the solution set $\sum(f, X^0)$ as

$$\sum(f, X^0) := \{x: f(x) = 0, x \in X^0\}. \quad (7)$$

Lemma 4.11. Let W_k be as in (5), and $\sum(f, X^0)$ as in (7). Then, $\bigcap_1^\infty W_k \subseteq \sum(f, X^0)$.

Proof. Let $x \in W_k \forall k$. It is to be shown that $x \in \sum(f, X^0)$. Since $x \in W_k \forall k$, it follows first that, for any k , an item X'_k must occur in the list $\mathcal{L}_k$ such that $x \in X'_k$. That is,

$$x \in X'_k \text{ for all } k. \quad (8)$$

By Lemma 4.9.

$$w(X'_k) \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (9)$$

From (8), (9), and Definition 2.7 it follows that the sequence $(X'_k)_{k=1}^\infty$ tends to x :

$$X'_k \rightarrow x \text{ as } k \rightarrow \infty.$$

Since f is continuous by Assumption 1, this gives

$$\bar{f}(X'_k) \rightarrow f(x) \text{ as } k \rightarrow \infty.$$

By the inclusion property in Theorem 2.2,

$$f(x) \in \bar{f}(X'_k) \subseteq F(X'_k). \quad (10)$$

Moreover, from (9) and Lemma 4.10

$$w(F(X'_k)) \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (11)$$

From (10), (11), and Definition 2.7

$$F(X'_k) \rightarrow f(x) \text{ as } k \rightarrow \infty. \quad (12)$$

Further, since X'_k is not yet discarded

$$0 \in F(X'_k) \text{ for any } k. \quad (13)$$

From (11), (13)

$$F(X'_k) \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (14)$$

Comparing (12) and (14) gives

$$f(x) = 0.$$

That is, $x \in \sum(f, X^0)$. $\square$

Lemma 4.12. Let U_k be as in (5). Then, $\bigcap_1^\infty U_k \subseteq \mathcal{G}$.

Proof. Consider the k th list $\mathcal{L}_k$ generated by the algorithm. Denote the i th box of this list as $X_{ki} = (Y_{ki}, \Lambda_{ki})$. From (5),

$$\begin{aligned} W_k &= \bigcup_{X_{ki} \in \mathcal{L}_k} X_{ki} = \bigcup_{Y_{ki}, \Lambda_{ki} \in \mathcal{L}_k} (Y_{ki}, \Lambda_{ki}) \\ &= \left(\bigcup_{Y_{ki} \in \mathcal{L}_k} Y_{ki}, \bigcup_{\Lambda_{ki} \in \mathcal{L}_k} \Lambda_{ki} \right). \end{aligned}$$

Therefore, $W_k = (U_k, V_k)$. This gives (by Remark 4.1, the intersections are non-empty)

$$\bigcap_1^\infty W_k = \bigcap_1^\infty (U_k, V_k) = \left(\bigcap_1^\infty U_k, \bigcap_1^\infty V_k \right). \quad (15)$$

From (7),

$$\begin{aligned} \sum(f, X^0) &= \{x: f(x) = 0, x \in X^0\} \\ &= \{(y, \lambda): y = h(\lambda), y \in Y^0, \lambda \in \Lambda^0\} \\ &= \{(h(\lambda), \lambda), \lambda \in \Lambda^0\}. \end{aligned} \quad (16)$$

By Lemma 4.11, $\bigcap_1^\infty W_k \subseteq \sum(f, X^0)$. From (15) and (16),

$$\begin{aligned} &\left(\bigcap_1^\infty U_k, \bigcap_1^\infty V_k \right) \\ &\subseteq \{(h(\lambda), \lambda), \lambda \in \Lambda^0\} \Rightarrow \bigcap_1^\infty U_k \\ &\subseteq \{h(\lambda), \lambda \in \Lambda^0\} = \bar{h}(\Lambda^0) =: \mathcal{G}. \quad \square \end{aligned}$$

Lemma 4.13. Let U_k be as in (5). Then, $d(U_k, \mathcal{G}) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. By Lemma 4.8, the unions U_k form a chain

$$U_1 \supseteq U_2 \supseteq U_3 \cdots. \quad (17)$$

From Lemmas 4.7 and 4.12

$$\mathcal{G} = \bigcap_1^\infty U_k. \quad (18)$$

Now, (17) and (18) together imply that the unions U_k converge to $\mathcal{G}$, i.e., $d(U_k, \mathcal{G}) \rightarrow 0$ as $k \rightarrow \infty$. $\square$

The following theorem gives the convergence property of the algorithm.

Theorem 4.14. The collection of solution rectangles Y (generated in the list $\mathcal{L}$ of the algorithm) converges, from without, to the template $\mathcal{G}$.

Proof. Let $A, B \in \mathfrak{R}^2$ be any two non-empty compact subsets. From Definition 2.6, it is clear that $d(A, B) = 0$ only if $A = B$. Therefore, it follows from Lemma 4.13 that convergence of unions U_k to the template $\mathcal{G}$ via d , i.e., $d(U_k, \mathcal{G}) \rightarrow 0$ as $k \rightarrow \infty$, means that U_k converges to $\mathcal{G}$. That this convergence is from *without* follows from the property $\mathcal{G} \subseteq U_k$, for any k (cf. proof of Lemma 4.7). $\square$

4.2. Finite termination

It is next shown that for $\varepsilon_x > 0$, the algorithm terminates in a finite number of iterations.

Theorem 4.15. *The algorithm terminates after at most $\chi^n - 1$ iterations, where $\chi = w(X^0)/\varepsilon_x$.*

Proof. First note that, after n, p successive bisections, where $n \geq 2$ (cf. Definition 3.3), $w(X) \leq 2^{-p}w(X^0)$. Thus, after at most $\Xi := n \log_2(w(X^0)/\varepsilon_x)$ successive bisections the processing of a subbox is completed (because then $w(X) \leq \varepsilon_x$). The algorithm produces a binary tree whose nodes are the regions obtained through successive bisections. The root of this tree is the node corresponding to the original region X^0 . The depth of this tree is equal to the maximum number of successive bisections, which is at most Ξ . The maximum total number of bisections occurs for a balanced tree, and is equal to the number of internal nodes in this tree. From Tucker (1980), this maximum number would be equal to $2^\Xi - 1 = \chi^n - 1$, where $\chi = w(X^0)/\varepsilon_x$. $\square$

Theorem 4.16. $\mathcal{G} \subseteq \mathcal{G}^{\text{alg}}$.

Proof. Let $\mathcal{L}_k^{\text{sol}}$ denote the list $\mathcal{L}^{\text{sol}}$ present at the end of the k th iteration of the algorithm. By Theorem 4.15, for a given $\varepsilon_x > 0$, the algorithm terminates at a finite iteration number, denoted α . Noting that for $\varepsilon_x > 0$ case, Y is entered when appropriate into $\mathcal{L}^{\text{sol}}$ in step 4c, and proceeding similarly as in the proof of Lemma 4.7, it can be shown that $\mathcal{G} \subseteq (\cup_{Y_i \in \mathcal{L}_{\alpha-1}^{\text{sol}}} Y_i) \cup U_k$, for any k . Combining this with the fact that the algorithm terminates when $\mathcal{L}$ is found empty at the end of the α th iteration, gives $\mathcal{G} \subseteq \cup_{Y_i \in \mathcal{L}_\alpha^{\text{sol}}} Y_i =: \mathcal{G}^{\text{alg}}$. $\square$

4.3. Safety and reliability

Definition 4.3 (Safe template). Let ω be a fixed frequency, and $g(s, \lambda)$ be as given above. Let $\mathcal{G}$ denote the actual template, and $\mathcal{G}^{\text{alg}}$ denote the template generated by an appropriate algorithm, both at ω . Then, $\mathcal{G}^{\text{alg}}$ is said to be a *safe template* if $\mathcal{G} \subseteq \mathcal{G}^{\text{alg}}$. Else, $\mathcal{G}^{\text{alg}}$ is said to be an *unsafe template*.

A *safe template*, therefore, includes the frequency responses of the entire plant set at the same ω . This inclusion property of a safe template guarantees that the resulting control design would be valid over the entire plant set. This important property is shown to hold for the template generated by the proposed algorithm:

Theorem 4.17 (Safety). *Let ω be a given frequency. Then, the generated template $\mathcal{G}^{\text{alg}}$ is a safe template as per Definition 4.3.*

Proof. From Theorem 4.16, $\mathcal{G} \subseteq \mathcal{G}^{\text{alg}}$. Therefore, by Definition 4.3, $\mathcal{G}^{\text{alg}}$ is a safe template. $\square$

In the developments given above, exact interval arithmetic was assumed in computing the results. Exact

arithmetic assumes infinite computing precision, and so is an *idealized* arithmetic. When such an arithmetic is implemented on a computer, then the endpoints of the intervals must be approximated by floating point or machine numbers. Each floating point operation is subject to error, and the default rounding may result in loss of rigor. Therefore, any theoretical results may be found invalidated in practice (on computers). An error analysis to account for these effects is rather troublesome.

Machine interval arithmetic provides a framework to generate reliable results taking into account all kinds of computational errors. In computations using machine interval arithmetic, values of input parameters are outwardly rounded to become intervals containing the true value. In addition, using outwardly rounded interval arithmetic throughout the computations assures that the computed result contains the actual range despite the presence of rounding errors. This outward rounding process is used together with the exact interval arithmetic rules to define a *machine* interval arithmetic.

Machine interval arithmetic can provide mathematically rigorous results from floating point operations on computers (Moore, 1979). The reader is referred to Klatte, Kulisch, Neaga, Ratz and Ullrich (1993) for details of machine interval arithmetic. Its utility for computing reliable results is evident from the following

Theorem 4.18 (Inclusion property of machine interval arithmetic (Moore, 1979)). *Let f be a real function of $x \in \mathfrak{R}^n$, and let $f(x)$ be an expression for f . Assume that the interval extensions $F(X)$ and $F^\diamond(X)$ (obtained using exact and machine interval arithmetic) are well-defined for the interval X corresponding to x . Then, it follows that $\bar{f}(X) \subseteq F(X) \subseteq F^\diamond(X)$, where $\bar{f}(X)$ denotes the range of f over X .*

Definition 4.4 (Reliable template). A template generated by an appropriate algorithm is said to be *reliable*, if it has been computed taking into account all kinds of computational errors, such as round-off, truncation, and approximation errors.

In practice, a machine interval arithmetic implementation of the proposed algorithm is used to account for all kinds of computational errors. The below theorem shows the reliability of the template results generated by a machine interval arithmetic implementation of the proposed algorithm.

Theorem 4.19 (Reliability). *Let ω be a given frequency. Let $\mathcal{G}^{\text{alg}}|_{\text{mia}}$ denote the template generated by a machine interval arithmetic implementation of the algorithm. Then, $\mathcal{G} \subseteq \mathcal{G}^{\text{alg}} \subseteq \mathcal{G}^{\text{alg}}|_{\text{mia}}$.*

Proof. Follows from Theorems 4.16 and 4.18. $\square$

5. Illustrative example

The following example demonstrates the capability of the algorithm to deal with a wide class of transfer functions. Consider the following transfer function:

$$g(s, \lambda) = \frac{(\lambda_1 + \lambda_4)e^{-s\lambda_2}}{\log_{10}\lambda_2 - e^{-s\lambda_1}/(1 + s \cos \lambda_4)} + \frac{1 - e^{-s\lambda_3}}{s\lambda_3(1 + (s\lambda_3/4\pi)^2)}$$

where the box of fundamental system parameters is given by $\lambda_1^0 \in [9.25, 9.35]$, $\lambda_2^0 \in [3, 5]$, $\lambda_3^0 \in [0.5, 0.7]$, and $\lambda_4^0 \in [0.49, 0.5]$. The template of the above uncertain transfer function is required at $\omega = 0.5$. For the purpose of computations, the algorithm is implemented in the programming language PASCAL-XSC (Klatte et al., 1993). With the choice $\varepsilon_x = 0.005$, the algorithm generated 7481 solution (angle-magnitude) rectangles covering the template, in about 10 min on a PC. For comparison purposes, a finite point approximation to the 'actual' template is also generated using MATLAB and simple gridding, with an 8-point grid for each parameter. For clarity, only the boundary of the template generated using the proposed algorithm is plotted versus the MATLAB template in Fig. 1. The template generated using the algorithm is seen to be quite satisfactory.

6. Conclusions

An algorithm has been proposed for generation of frequency response templates for the wide class of magnitude and phase functions that are *continuous* with respect

Fig. 1. The boundary of the template generated using the proposed algorithm. The inner points are those generated using MATLAB and simple gridding.

to the uncertain system parameters. Other key aspects of the algorithm are that the generated templates are of *arbitrary* accuracy, *safe*, and *reliable*. An illustrative example has been provided.

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