Abstract

In this paper, we examine the experimental data of Kuhlthau [A.R. Kuhlthau, Air friction on rapidly moving surfaces, Journal of Applied Physics 20 (1949) 217] on two concentric rotating cylinders with a low pressure gas in the gap. The data are used to deduce the slip coefficient in the slip flow and transition flow regimes ($0.04 \leq Kn \leq 8.3$). The obtained values of tangential momentum accommodation coefficient for dry air is 1.13 in the slip regime and 1.70 in the transition regime. These results suggest a dependence of slip coefficient on Knudsen number, $Kn$. Some comments on velocity profile inversion reported in the literature for such flows and calculation of slip velocity on non-planar surfaces are also made. The results are important because they suggest that the Navier–Stokes equation, with slight correction, can be used for analysis of flow in such high Knudsen number regimes.

Keywords: Rarefied gas; Tangential momentum accommodation coefficient; Slip coefficient; Slip regime; Transition regime

1. Introduction

The Couette flow problem, whereby fluid between two long parallel plates is dragged due to uniform motion of one plate while the other plate is held stationary, is one of the simplest problems in fluid mechanics. The simplicity arises partly because the non-linear terms in the Navier–Stokes equations drop-out. The equivalent of this problem in polar coordinates is two concentric rotating cylinders. In this paper, we consider a special case of low pressure gas between the cylinders, such that the mean free path of the gas ($\lambda$) is comparable to the gap ($R_2 - R_1$) between the cylinders, implying that the Knudsen number ($Kn = \lambda/(R_2 - R_1)$) is large. Here $R_1$ and $R_2$ are the radius of the two cylinders.

It is well known that the no-slip boundary condition is not satisfied under rarefied flow condition – rather the gas slips at the cylinder walls. Millikan [1] postulated that this slipping behaviour is due to "molecular inhomogeneities", while Cao et al. [2] argue that the mechanism for momentum transfer between the wall and gas is due to trapping of gas molecules in the potential wells of the surfaces. The gas molecules may undergo several collisions and may escape after a residence time, during which time momentum exchange between the fluid and wall is accomplished [2]. The slip coefficient (which is related to slip velocity, through Eq. (4) presented later) can be deduced by measuring the torque on the stationary cylinder. This idea of measuring the slip coefficient ($C_1$) or equivalently the tangential momentum accommodation coefficient (TMAC or $\sigma$) goes back to Millikan [1]. A knowledge of slip coefficient is particularly important for modeling of flow in the slip ($10^{-3} \leq Kn \leq 10^{-1}$) and transition ($10^{-1} \leq Kn \leq 10$) flow regimes. The problem is relevant in the present scenario because of interest in studying flow, mixing and other aspects of gas flow in microchannels, where the Knudsen number is in the above mentioned range [3–6].

The rotary Couette flow problem has also proved a testing ground for the generalized slip model of Einzel et al. [7]. The model when applied to rarefied gas [8] with the inner cylinder rotating and the outer cylinder held stationary,
predicts an inverted velocity profile for low values of \( \sigma \) \((\sigma < 0.1)\) \([9,10]\). An inverted velocity profile implies that the velocity of the gas is larger than the rotating cylinder and is because of slipping of the gas at the walls. The observation is supported by the direct simulation Monte Carlo (DSMC) data of Aoki et al. \([9]\). Yuhong et al. \([10]\) show that velocity inversion occurs even when the TMAC on the two cylinders are assumed equal. It is however not clear if such low values of TMAC can actually occur.

At present there seems to be enough interest in determining the TMAC \([4,11]\). This along with the intriguing phenomena of velocity inversion motivated us to look for any experimental data available in the literature. The excellent data of Kuhlthau \([12]\) in the Knudsen number range of 0.04–8.3 seems to be appropriate for the present objective and is therefore re-analyzed in the paper. This data in conjunction with the Navier–Stokes equations, provide a good estimate of TMAC (or slip coefficient). It is also argued, based on Kuhlthau’s data, that the applicability of the Navier–Stokes equations can be extended by appropriately tuning the value of the slip coefficient, in support of a similar conclusion drawn by Dongari et al. \([13]\).

### 2. Experimental setup of Kuhlthau \([12]\)

The experimental setup of Kuhlthau \([12]\) consisted of a rotating inner cylinder (radius \( R_1 = 5 \) cm) and stationary outer cylinder (radius \( R_2 = 6.25 \) cm). The length of the inner cylinder \((L)\) was 5 cm, and is machined from a solid block of forged Duralumin alloy ST-14. The axial gap between the rotating inner cylinder and outer cylinder was also 1.25 cm. The angular velocity of the rotor \((\omega)\) was varied between 400 and 1700 revolutions per second and the resulting torque was measured precisely by means of a coaxial phosphor–bronze wire, of 373 \(\mu\)m diameter.

Dry air was used as the working fluid. The pressure was varied between 0.48 and 100 torr \((0.064–13 \) Pa) in the experiments, while the temperature was held constant at 26 \(^\circ\)C. The setup and measurement tools were validated by comparing the experimental data against predictions from free-molecular theory. An excellent agreement was obtained at \( Kn = 8.3 \); the agreement remains good (maximum deviation of 3%) at \( Kn = 4 \), but differences become unacceptably large at lower values of Knudsen numbers, suggesting that the free-molecular theory is applicable for \( Kn \geq 4 \) but not for smaller values, at least for this problem.

Here, analysis of the data is performed by starting from the other (continuum) end. The solution of the Navier–Stokes equations for circular Couette flow with slip at the cylinders, is therefore briefly reviewed in the following section.

### 3. Governing equations

The equations of motion in the radial and tangential directions for the problem under consideration are \([14]\):

\[
-\frac{u^2}{r} = -\frac{1}{\rho} \frac{d\rho}{dr},
\]

\[
0 = \mu \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (ru) \right],
\]

where \( u \) is tangential velocity, \( \rho \) is pressure, \( r \) denotes radial coordinate, and \( \mu \) and \( \rho \) are, respectively, viscosity and density of the gas. Because we are not interested in the solution for pressure, only Eq. (2) will be considered in this paper. The boundary conditions at the two cylinders are:

\[
u_g - w_w = \omega R_1 + \frac{2 - \sigma}{\sigma} \lambda \left( \frac{du}{dr} - \frac{u}{r} \right)_{w} \text{ at } r = R_1,
\]

\[
u_g - w_w = -\frac{2 - \sigma}{\sigma} \lambda \left( \frac{du}{dr} - \frac{u}{r} \right)_{w} \text{ at } r = R_2,
\]

where the subscripts ‘g’ and ‘w’ refer to gas and wall, respectively. As pointed out by Yuhong et al. \([10]\), the last

### Nomenclature

- \( A \): see Eq. (8)
- \( B \): see Eq. (9)
- \( C_1 \): first-order slip coefficient \((-)\)
- \( C_1^\text{av} \): average first-order slip coefficient \((-)\)
- \( C_2 \): second-order slip coefficient \((-)\)
- \( C_p \): correction factor \((-)\)
- \( Kn \): Knudsen number \((-)\)
- \( L \): length of the inner cylinder \((\text{m})\)
- \( \rho \): pressure \((\text{Pa})\)
- \( r \): radial coordinate \((\text{m})\)
- \( R_1 \): radius of the inner cylinder \((\text{m})\)
- \( R_2 \): radius of the outer cylinder \((\text{m})\)
- \( T \): torque \((\text{N m})\)
- \( u \): tangential velocity \((\text{m/s})\)
- \( \lambda \): mean free path of gas \((\text{m})\)
- \( \mu \): viscosity of fluid \((\text{Pa s})\)
- \( \omega \): angular velocity \((\text{rad/s})\)
- \( \rho \): density of fluid \((\text{kg/m}^3)\)
- \( \sigma \): tangential momentum accommodation coefficient \((-)\)
- \( \bar{\sigma} \): average tangential momentum accommodation coefficient \((-)\)
- \( \tau_i \): tangential momentum of incident molecules \((\text{kg m/s})\)
- \( \tau_r \): tangential momentum of reflected molecules \((\text{kg m/s})\)
- \( \text{TMAC} \): tangential momentum accommodation coefficient
- \( \text{DSMC} \): direct simulation Monte Carlo

\[u\text{ predicts an inverted velocity profile for low values of } \sigma \text{ \((\sigma < 0.1)\) \[9,10\]. An inverted velocity profile implies that the velocity of the gas is larger than the rotating cylinder and is because of slipping of the gas at the walls. The observation is supported by the direct simulation Monte Carlo (DSMC) data of Aoki et al. \[9\]. Yuhong et al. \[10\] show that velocity inversion occurs even when the TMAC on the two cylinders are assumed equal. It is however not clear if such low values of TMAC can actually occur.

At present there seems to be enough interest in determining the TMAC \[4,11\]. This along with the intriguing phenomena of velocity inversion motivated us to look for any experimental data available in the literature. The excellent data of Kuhlthau \[12\] in the Knudsen number range of 0.04–8.3 seems to be appropriate for the present objective and is therefore re-analyzed in the paper. This data in conjunction with the Navier–Stokes equations, provide a good estimate of TMAC (or slip coefficient). It is also argued, based on Kuhlthau’s data, that the applicability of the Navier–Stokes equations can be extended by appropriately tuning the value of the slip coefficient, in support of a similar conclusion drawn by Dongari et al. \[13\].

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Here, analysis of the data is performed by starting from the other (continuum) end. The solution of the Navier–Stokes equations for circular Couette flow with slip at the cylinders, is therefore briefly reviewed in the following section.

### 3. Governing equations

The equations of motion in the radial and tangential directions for the problem under consideration are \[14\]:
term in Eqs. (3) and (4) arises because the slip velocity is actually related to the shear stress at the wall and not just the velocity gradient. Note that this last term has often been neglected with slip flow on non-planar surfaces [10]. It can be easily shown that these terms make a significant contribution to the solution for the present problem. Usually $\sigma$ is written in terms of slip coefficient, $C_1$ (e.g. [3]), where
\[ C_1 = \frac{2 - \sigma}{\sigma} \quad (5) \]
and $\sigma$ can be related to the tangential momentum of the incident ($\tau_i$) and reflected ($\tau_r$) molecules using the following relation:
\[ \sigma = \frac{\tau_r - \tau_i}{\tau_i} \quad (6) \]

Note that the second-order slip coefficient, $C_2$ [15,3,13], does not exist for this problem, because the derivative of shear stress at the wall is identically zero. While this represents a substantial simplification in processing the data, the value of $C_2$ which is also of considerable significance and difficult to measure, cannot be determined from this flow.

On solving Eq. (2) with the help of Eqs. (3)–(5), the following solution is obtained [10]:
\[ u = \frac{\omega}{A - B} \left( Ar - \frac{1}{r} \right), \quad (7) \]
where
\[ A = \frac{1}{R_2} \left( 1 - C_1 \frac{2\lambda}{R_2} \right), \quad (8) \]
\[ B = \frac{1}{R_1} \left( 1 + C_1 \frac{2\lambda}{R_1} \right). \quad (9) \]
The torque ($T$) on the inner cylinder is therefore
\[ T = \frac{4\pi \mu \omega L}{B - A}, \quad (10) \]
Note that the torque acting on the top and bottom surfaces (or the two flat surfaces) of the inner cylinder, have not been accounted for in the above equation. Due to the relatively short length of the rotating inner cylinder and comparable gap thickness on all its sides, the correction to the above torque equation may be substantial. For example, Muralidhar and Biswas [14] suggest that the length of the cylinder be chosen such that less than 5% of the total torque experienced by the cylinder is contributed by the ends. This criteria was apparently overlooked by Kuhlthau [12] while designing his experimental setup; the length to diameter ratio of the rotor was maintained less than unity in order to obtain enhanced rotor stability. We therefore rewrite Eq. (10) with a correction factor ($C_F$) as
\[ T = \frac{4\pi \mu \omega L}{B - A} \quad (11) \]
The second purpose of using a correction factor is to see if the match between experimental data and Eq. (11) can be improved for very high values of Knudsen numbers, for which the data is available.

Note that the torque increases linearly with angular velocity, or
\[ (B - A) \frac{dT}{d\omega} = 4\pi \mu C_F \quad (12) \]
with the slope dependent on pressure and slip coefficient. Therefore, for a given pressure the value of $C_1$ can be deduced from the following relationship:
\[ C_1 = \left[ \frac{dT}{d\omega} \left( \frac{1}{R_2^2} - \frac{1}{R_1^2} \right) + 4\pi \mu L C_F \right] \left[ \frac{dT}{d\omega} (2\lambda) \left( \frac{1}{R_1^2} + \frac{1}{R_2^2} \right) \right]^{-1}, \quad (13) \]
which is obtained by algebra starting at Eq. (12). The derivative of torque with respect to angular speed as determined by the data of Kuhlthau [12], at different pressures has been tabulated in Table 1. Note that Eq. (13) has been used in the following section to obtain the value of $C_1$ from his dataset.

4. Data analysis

Although there is only one equation (Eq. (13)) with two unknown parameters ($C_1$ and $C_F$), both of them can be obtained by regressing the experimental data. The best results (as determined by minimizing the difference between the two sides of Eq. (12)) are obtained for $C_F = 1.91$ and $C_1 = 1.67$. The value of the two sides of Eq. (12), along with the percentage difference, is tabulated in Table 2.

Table 3 shows that with this value of correction factor, a more-or-less constant value of TMAC is obtained (i.e. barring two outliers in 10 points) in the transition regime (see also Fig. 1), while a value close to unity is suggested with a large scatter in the slip regime. Fig. 1 also shows that in the slip flow regime, the TMAC can be greater than unity, suggesting that there is back scattering of molecules. The average value of $\sigma$ in the transition regime (after removing the

<table>
<thead>
<tr>
<th>$p$ (µm Hg)</th>
<th>$\frac{dT}{d\omega}$ (N m s⁻¹)</th>
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</thead>
<tbody>
<tr>
<td>0.48</td>
<td>5.51 × 10⁻⁶</td>
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<td>1</td>
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<tr>
<td>5</td>
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<tr>
<td>10</td>
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</tr>
<tr>
<td>100</td>
<td>1.49 × 10⁻⁷</td>
</tr>
</tbody>
</table>
The corresponding values of $C_1$ in the two regimes are 1.70 and 1.13, respectively.

Although the use of an arbitrary correction factor ($C_F = 1.91$) looks unsatisfactory, as mentioned above, this factor can be used to ‘match’ experimental data against predictions from the Navier–Stokes equations. This matching is required because, as is well known, the continuum assumption on which the Navier–Stokes equations are based is not valid at high Knudsen numbers (especially in the transition and free-molecular regimes). It is noted that the Navier–Stokes equations are correct to the first order in terms of Knudsen number and the correction factor, in a sense, accounts for the higher order terms that have been otherwise neglected from the governing equations. The above results suggest that these equations, however inappropriate, can be extended to higher Knudsen numbers with an appropriate correction factor. It is reassuring that a single correction factor is able to match the experimental data with theory over a sufficiently large range of Knudsen number (0.04–8.3). On the other hand, we recognize that the same correction factor will not work for all cases and determining it may not always be straightforward. The correction factor approach has recently been proposed by Dongari et al. [13] for the analysis of rarefied gas flow in channels. They showed that the results from theory based on continuum assumptions can be matched with that from the Boltzmann equation for a sufficiently large range of Knudsen numbers by choosing an artificial set of slip coefficients. Note that the value of $C_1 = 1.87$ proposed by Dongari et al. [13] agrees decently well with the value of $C_1 = 1.70$ for large Knudsen numbers found herein.

5. Discussion

The values of slip coefficient suggested in the literature [13] is tabulated in Table 4. Although the value of the second slip coefficient is also provided in the table for the sake of completeness, $C_2$ cannot be determined for the present flow, as discussed above. It is apparent from the table that outliers) is 0.74, and that for the slip regime is 0.94. The corresponding values of $C_1$ in the two regimes are 1.70 and 1.13, respectively.

![Fig. 1. Variation of tangential momentum accommodation coefficient with Knudsen number.](image)
most of the previous approaches have been theoretical. Note that these values have been derived without a consideration of the Knudsen number dependency of the slip coefficients. Second, there is a gross disagreement in the values of the slip coefficients, in particular the suggested value of $C_2$ varies over an order of magnitude. The latter observation underscores the importance of obtaining reliable experimental data in different geometries, over a wide range of Knudsen numbers.

For the plane Couette flow, Chiang’s data (see [22]) with an unknown gas in the Knudsen number range of 0.02–5 in conjunction with an analysis similar to that performed here, suggest that any $\sigma$ between 1 and 0.8 (and perhaps lower) will fit the data well in the lower range of Knudsen numbers ($Kn < 0.05$). However, $\sigma = 0.9$ (or $C_1 = 1.22$) seems to give a superior fit to experimental data of Chiang and at higher Knudsen numbers ($0.05 < Kn < 5$). This suggests a dependence of slip coefficient on Knudsen number.

A dependence of slip coefficient on Knudsen number is also supported by the experimental data of Sreekanth [15] for flow of nitrogen in a tube. Sreekanth [15] had noted that comparison between his experimental data and the theory derived by him improves by changing the values of $C_1$ and $C_2$, from 1 and 0 for $Kn \leq 0.03$, to 1.1466 and 0 for $0.03 < Kn < 0.13$, and to 1.1466 and 0.14 for $Kn \geq 0.13$. Maurer et al. [11] measured the slip coefficient for flow of nitrogen and helium in a microchannel, with Knudsen number extending up to 0.8 for helium and 0.6 for nitrogen. The values of TMAC are 0.91 ± 0.03 for helium and 0.87 ± 0.06 for nitrogen. The present analysis on Kuhlthau’s data suggest $C_1 = 1.13$ for $Kn < 0.1$ and $C_1 = 1.70$ for $0.1 < Kn < 8.3$, which compares reasonably well with the earlier values.

Based on the above observation, we expect the average value of $\sigma$ in the free-molecular regime to be lower than 0.74; however, its precise value cannot be determined from the present dataset. An explanation for the decrease in the value of $\sigma$ with an increase in Knudsen number does not seem to be available in the literature. We conjecture that the residence time of the gas molecules trapped in the potential wells of the surfaces reduces with an increase in Knudsen number, leading to reduced momentum exchange between the fluid and wall. Therefore, the percentage of molecules being diffusively reflected decreases and the value of TMAC decreases. The above mentioned observation is also consistent with the notion of slipping increasing with an increase in molecular inhomogenities.

The above data analysis and literature survey allows us to comment on the possibility of experimentally realizing a velocity inversion, which is rather non-intuitive [9,10]. We note that, although the value of thermal accommodation coefficient for helium on tungsten or platinum is very small (of the order of 0.05 [23]), the experimental data surveyed in this paper do not support a TMAC of 0.1 or lower. We therefore remark that, although a velocity inversion is theoretically possible, it would probably not be experimentally attainable.

We now comment on the effect of neglecting the $u/r$ term in Eqs. (3) and (4), as is usually done [10]. It can be easily verified that neglecting the $u/r$ term for the present problem would lead to unphysical torque values (in particular, a reversal in the sign for torque occurs). Therefore, retaining these terms is important while calculating the slip velocity at the cylinder walls.

6. Conclusions

The experimental data of Kuhlthau [12] for rarefied gas flow between a rotating/stationary cylinder pair, has been re-analyzed in this paper with the aim of determining the slip coefficients. The slip coefficient is found to be 1.13 and 1.70 in the slip and transition regimes, respectively, suggesting a dependence of slip coefficient on Knudsen number. The corresponding values of the tangential momentum accommodation coefficient in the two regimes are 0.94 and 0.74, respectively. However, the second coefficient of slip cannot be determined from this flow.

These results are important from the point-of-view of numerical simulation of rarefied gases. Because of the difficulty in making precise measurements, simulation should be the more commonly used approach to study gas flows through microchannels. The match between theory and experimental data suggest that it should be possible to simulate gaseous flow in a large range of Knudsen number with the Navier–Stokes equations itself (with appropriate correction factors). Once firmly proven (i.e. for various geometries and other parameters), this would constitute a great simplification considering that simulation and analysis tools are perhaps better developed for solving the Navier–Stokes equation as compared to the Boltzmann equation.

References


